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Lectures on the Mathematical Method in Analytical Economics

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PART A

The Leontief Model and the Technological Basis of Production

LECTURE 1

Introduction and Outline

1. What Will and What Will Not Be Treated

Mathematical economics currently includes, and perhaps is even dominated by, a number of branches with which we will have little to do. Thus, in order to define the subject of the present lectures, it is well to say something about these excluded branches. One topic that we shall not discuss to any great length is the subject that might be called *efficiency economics* in general, and is often called by the several names of its principal techniques—*linear programming*, *operations research*, perhaps also *theory of games*. In these subjects, the aim is to find the optimal adjustment, in one or another sense, to a given situation; they refer with greatest cogency and success to the profit-making possibilities of a single firm. As an omnibus reference to this area of thought let me cite Vajda's *Linear Programming and the Theory of Games*, and also von Neumann and Morgenstern's sparkling *Theory of Games and Economic Behavior*. Nor will we deal with *econometrics*, i.e. applied and theoretical economic statistics, except incidentally. Instead, we shall take economics as the cognitive study of a given object, the economy, and ask in the sense of natural science: what is this object like, how does it behave, and why? For this reason, we find the term *analytical* prefacing *economics* in our title. In spirit, our economics will be theoretical or speculative rather than directly empirical, and thus close in its basic approach to what has been called classical economics. In form, however, we will be more systematically mathematical. The branch of mathematics of which we will make greatest use will be the theory of matrices; let me here make reference to D. T. Finkbeiner's *Intro-*

duction to Matrices and Linear Transformations, to Paul Halmos' *Finite Dimensional Vector Spaces*, Gantmacher's *Theory of Matrices*, and note the existence of numerous other introductory works on this subject. From time to time we will use a bit of calculus.

We will begin with a discussion of the theory of equilibrium prices—what has been traditionally called value theory—and go on to a discussion of business cycle theory, beginning with a model like that introduced by Lloyd Metzler, and developing the connection between this cycle theory and the equilibrium analysis that is more commonly called Keynesian. In the economic literature let me cite, in the first place, the famous *General Theory* of Keynes, which, as a pioneering work of science, is worth studying in spite of its numerous pedagogical and even theoretical mare's nests. A stimulating companion volume for the admirer of Keynes is Henry Hazlitt's *The Failure of the New Economics. An Analysis of the Keynesian Fallacies.* A superior mathematical exposition of the Keynesian theories is K. Kurihara's *Introduction to Keynesian Dynamics;* another, particularly fine, work of a similar sort is H. J. Brems's *Output, Employment, Investment.* Much of what we have to say will make reference to the "input-output" model of W. Leontief, on which there exists a vast literature. A good sample of this literature, full of references, is *Activity Analysis of Production and Allocation*, T. C. Koopmans, ed. Our attempts to compare speculative results with economic reality will be enormously facilitated by the extensive and painstaking work of the *National Bureau of Economic Research*, published in the form of a great many separate studies. A very fresh and stimulating empirical account of business cycles is the easily available *Business Cycles and their Causes* by W. C. Mitchell.

2. A Bouquet of Warnings

Mathematics may perhaps have a valuable role to play in economics—but its application brings several dangers. Mathematics necessarily works with exact models. In the course of investigating such a model, it is easy to forget that the mathematical exactness of one's reasoning has nothing to do with the exactness with which the model reflects economic reality. For this reason, a few dampening admonitions are in order. I quote the first and most severe from Ludwig von Mises' *Human Action:*

The problems of prices and costs have been treated also with mathematical methods. There have even been economists who held that the only appropriate method of dealing with economic problems is the mathematical method and who derided the logical economists as "literary" economists.

If this antagonism between the logical and the mathematical economists were merely a disagreement concerning the most adequate procedure to be applied in the study of economics, it would be superfluous to pay attention to it. The better method would prove its preeminence by bringing about better results. It may also be that different varieties of procedure are necessary for the solution of different problems and that for some of them one method is more useful than the other.

However, this is not a dispute about heuristic questions, but a controversy concerning the foundations of economics. The mathematical method must be rejected not only on account of its barrenness. It is an entirely vicious method, starting from false assumptions and leading to fallacious inferences. Its syllogisms are not only sterile; they divert the mind from the study of the real problems and distort the relations between the various phenomena.

The deliberations which result in the formulation of an equation are necessarily of a nonmathematical character. The formulation of the equation is the consummation of our knowledge; it does not directly enlarge our knowledge. Yet, in mechanics the equation can render very important practical services. As there exist constant relations between various mechanical elements and as these relations can be ascertained by experiments, it becomes possible to use equations for the solution of definite technological problems. Our modern industrial civilization is mainly an accomplishment of this utilization of the differential equations of physics. No such constant relations exist, however, between economic elements. The equations formulated by mathematical economics remain a useless piece of mental gymnastics and would remain so even if they were to express much more than they really do.

A corresponding sentiment is voiced by Keynes in his *General Theory:*

It is a great fault of symbolic pseudo-mathematical methods of formalizing a system of economic analysis, such as we shall set down in section VI of this chapter, that they expressly assume strict independence between the factors involved and lose all their cogency and authority if this hypothesis is disallowed; whereas, in ordinary discourse, where we are not blindly manipulating but know all the time what we are doing and what the words mean, we can keep "at the back of our heads" the necessary reserves and qualifications and the adjustments which we shall have to make later on, in a way in which we cannot keep complicated partial differentials "at the back" of several pages of algebra which assume that they all vanish. Too large a proportion of recent "mathematical" economics are mere concoctions, as imprecise as the initial assumptions they rest on, which allow the author to lose sight of the complexities and interdependencies of the real world in a maze of pretentions and unhelpful symbols.

A more optimistic if still cautious opinion is stated by Professor F. H. Knight in the 1954 *Britannica.*

Any brief statement of principles is bound to make economic theories appear thinner and more remote from the concrete facts of economic life than they are. There is a place and a need for all degrees of generality. In recent decades this need has found increasing expression in the developing and spreading study of mathematical economics, in which exposition is made accurate and compact by the use of graphs and of algebraic formulae.

Only by the use of mathematics is it possible to bring together into a single comprehensible picture the variety, the complexity, and most of all the interdependence of the numerous factors which determine prices, costs, output and demand and the wages or hire of productive agents. . . . The principal value of such elaborate and abstract systems lies in forcibly reminding the enquirer that a change in practically any economic variable has direct or indirect effects on innumerable other magnitudes, and so preventing him from fatally oversimplifying conceptions of economic cause and effect. . . .

The more theoretical parts of economics cannot be taken to be a complete and adequate account of the mechanism of modern economic life. They afford serviceable approximations to partial, but important aspects of the truth.

The most striking and possibly the most important characteristic of recent work in economics, as contrasted with the older, is its greater realism. It does not attempt to do without abstract conceptions, but it does attempt to take these from the world of affairs, or bring them into line with facts.

Hoping to approach Professor Knight's high goal, we may begin our investigations.

3. Introduction of a Model (Single Labor Sector)

By an economy we shall mean a complex of activities in which various commodities are produced and subsequently either consumed or utilized in the production of further commodities. If the economy absorbs commodities from outside itself, or if it supplies commodities to the outside, it is called *open.* On the other hand, if the economy is completely self-contained it is called *closed.* We wish to describe

a model of an economy. Our model will in the first instance be open, in that labor must be supplied to the system by a "household sector" and products must be supplied to the household sector by the system. The model will then be "closed" by introducing labor as an additional commodity which is "used up" in the production of various other commodities, and for the production of which these various other commodities are required.

After formulating our model, we shall first indicate the manner in which it gives rise to a simple but interesting theory of prices; next give a brief discussion of the extent to which the model is a faithful reflection of the real economy; and subsequently pass to an extended mathematical analysis of the model, and to an investigation of the question of what additional and useful relationships among the various parameters of an economy can be elucidated by using the model.

We begin by establishing, in some definite but entirely arbitrary way, a certain standard physical unit for each commodity, as, e.g. 1 car, 1 ton of coal, 1 bushel of wheat, etc., hereinafter called one unit of the commodity. The process of production of any commodity requires appropriate amounts both of *circulating* and of *fixed capital.* Thus, for instance, to produce one ton of pig iron it is required, in the first place, that certain amounts of coal—say, half a ton, and certain amounts of iron ore—say, one and a half tons, be used up; but, in addition it is required that a blast furnace be tied up, for a certain period, say for half a day. The blast furnace is *tied up* but not *used up,* and hence reckons only as fixed, but not as circulating capital.

These two aspects of production will be described in our general model as follows. Let the economy involve a total of n commodities, i.e. let $C_1, \cdots, C_n$ be a total list of the commodities produced in an economy; cars, cigarettes, typewriters, etc. To produce one unit of any given commodity C_i, it is (technologically) required that various amounts π_{ij} of other commodities C_j be used up; in addition, it is (technologically) required that ϕ_{ij} units of C_j be tied up (and thus not available for the manufacture of some other product) for a standard production period, say one day, even though these ϕ_{ij} units of C_j are not necessarily used up. Note that if the standard unit of C_1 is, say, a bushel, and that of C_2 is, say, one ton, π_{12} has the dimensions tons per bushel, and ϕ_{12} the dimensions ton-days per bushel.

ϕ_{ij} is said to be the amount of C_j *utilized* in the production of one unit of C_i, while π_{ij} is said to be the amount of C_j *consumed* in the production of one unit of C_i. When a commodity is *consumed* it is also *utilized* and therefore we shall assume

(1.1) $$\text{If } \pi_{ij} > 0 \quad \text{then} \quad \phi_{ij} > 0 \quad i,j = 1, \cdots, n.$$

The model as we have thus far defined it is called the *open Leontief model;* the matrix π_{ij} is often called the *input-output matrix,* and analysis of such a model is often called *input-output analysis.* The matrix ϕ_{ij} may be called the *fixed capital matrix.* It is clear upon a moment's reflection that this open model, as it has been defined, permits us to deduce, from a given "final demand" for a certain "bill of goods," what inputs are required; and thus, for instance, by considering the desired output of military goods in a wartime situation, to predict where "bottle-necks" are apt to develop. This sort of application has often been stressed; reference may be made to the work *Activity Analysis of Production and Allocation* cited above. Matrices π_{ij} for the American economy divided into fifty and into two hundred sectors have been computed by the Bureau of Labor Statistics. Our interest, however, will not be in this direct sort of "bottleneck analysis," but in the use of the input-output model as a framework for more abstract economic analysis. For this reason, we proceed at once to a description of a corresponding closed model.

To close our model, we must introduce labor as an input and as an output. Let π_{jo} denote the amount of labor (measured say, in man-hours) required for the production of C_j, and let π_{oj} be the amount of C_j which is "consumed in order to produce a man-hour of labor," i.e., the average real wages paid out per hour of labor. By the introduction of these matrix elements the model economy is rendered closed, i.e. the set of commodities produced is the same as the set of commodities utilized in production.

In our simple linear economic model (often called the closed Leontief model) we may readily set up a theory of prices. Let $p_0, p_1, \cdots, p_n$ be the prices of the various products produced; then $p_0, p_1, \cdots, p_n$ are also the prices of the commodities utilized and/or consumed in production.

We formulate the conditions that must be satisfied by the p_i.

The price of a commodity C_i is made up of the sum of two terms.

The first is the sum of the values of all of the products consumed in the manufacture of C_i, i.e.

(1.2) $$\sum_{j=0}^{n} \pi_{ij} p_j.$$

The second is the "return to capital," "markup," or "profit" proportional to the sum of the values of the products which are utilized but not necessarily consumed in the manufacture of C_i, i.e.

(1.3) $$\rho \sum_{j=0}^{n} \phi_{ij} p_j.$$

(We take $\phi_{io} = 0$).

We have here taken an essential step in assuming the rate of profit, ρ, to be the same for all types of production. This corresponds to the ordinary assumption, in the theory of prices, of "free competition"; it can be justified in the usual way by arguing that a situation in which the production of different commodities yields different rates of profit cannot be stable, since investments would be made only in the industry yielding the highest rate of profit to the exclusion of other commodities yielding lower rates of profit. Long-term equilibrium, of which our simple theory is alone descriptive, would be reached only when all such rates of profit became equal. The proportionality constant ρ has the dimensions per cent per day (or year).

Forming the price of C_i additively out of the two expressions (1.2) and (1.3), we have the set of equations

(1.4) $$p_i = \sum_{j=0}^{n} \pi_{ij} p_j + \rho \sum_{j=0}^{n} \phi_{ij} p_j, \quad i = 1, \cdots, n.$$

Thus far we have only an "open" system of equations for the prices p_i. We can obtain a closed system by recalling that π_{oj} denotes the collection of commodities which form real wages for an hour's labor; thus the price p_0 of an hour's labor must be given by the equation

(1.5) $$p_0 = \sum_{i=0}^{n} \pi_{oi} p_i.$$

If we introduce additional matrix elements ϕ_{oj} by putting $\phi_{oj} = 0$, we may write (1.5) in the same form as the equation (1.4), and hence may write (1.5) and (1.4) together in the simple form

$$(1.6)\qquad p_i = \sum_{j=0}^{n} \pi_{ij}p_j + \rho \sum_{j=0}^{n} \phi_{ij}p_j, \quad i = 0, \cdots, n.$$

This set of $n + 1$ equations is homogeneous in the $n + 1$ variables p_j, but contains the additional unknown ρ. Thus, we would expect that the system (1.6) determines the quantity ρ and the set of n ratios of the $n + 1$ quantities p_i. We will show in the next lecture that this is rigorously correct. Before going over to the necessary detailed and general mathematical investigation, however, let us examine some simple transformations and special cases of the system (1.6).

In the first place, we may make use of the particularly simple form of equation (1.5) to eliminate p_0 from the system (1.6). This gives

$$(1.7)\quad p_i = \sum_{j=1}^{n} [\pi_{ij} + (1 - \pi_{oo})^{-1}\pi_{io}\pi_{oj}]p_j + \rho \sum_{j=1}^{n} \phi_{ij}p_j, \quad i = 1, \cdots, n.$$

If we define a modified input-output and fixed capital matrix by

$$(1.8)\quad \tilde{\pi}_{ij} = \pi_{ij} + (1 - \pi_{oo})^{-1}\pi_{io}\pi_{oj}; \ \tilde{\phi}_{ij} = \phi_{ij}, \quad i,j = 1, \cdots, n,$$

the system (1.7) takes on the form

$$(1.9)\qquad p_i = \sum_{j=1}^{n} \tilde{\pi}_{ij}p_j + \rho \sum_{j=1}^{n} \tilde{\phi}_{ij}p_j,$$

i.e. takes on a form exactly analogous to that of the system (1.6) but with the price p_0 eliminated. The system (1.9) may in consequence be called the *labor-eliminated* form of our price equations, and the matrices $\tilde{\pi}_{ij}$ and $\tilde{\phi}_{ij}$ the *labor-eliminated input-output matrix* and the *labor-eliminated fixed capital matrix* respectively. The transformation which leads from (1.6) to (1.9) may be given the following heuristic interpretation. If each time labor appears as an input in production we replace this input by the corresponding real wage bill, we come to a hypothetical situation in which the only inputs required for the production of commodities are other (non-labor) commodities. Thus we may, if it is convenient for one or another theoretical purpose, consider our model to refer to a self-enclosed world of material commodities, produced out of each other with no additional input. We will make use of this modified description of our model economy (which is, of course, entirely equivalent to our initial description) at a number of points in what follows.

It is instructive to study the system (1.9) in its most trivial special case, the case $n = 1$, i.e. the case in which there is only one commodity. If there is but one commodity equations (1.9) become

$$(1.10)\qquad p = \tilde{\pi}_{11}p + \rho\tilde{\phi}_{11}p$$

with the solution

$$(1.11)\qquad \rho = (1 - \tilde{\pi}_{11})/\tilde{\phi}_{11}.$$

The "price" in this case has, of course, no significance, since our theory is one of relative prices, i.e. price ratios, only, the "absolute price level" being meaningless in a theory like the one before us. The rate of profit, however, is given by (1.11). The most significant observation we can make about this equation is that *the rate of profit, ρ, is positive if and only if $\tilde{\pi}_{11} < 1$*, i.e. if and only if we are able to produce one unit of commodity without consuming an equivalent or larger amount of commodity. It is clear that only in this case is production capable of yielding a physical surplus. The case $\tilde{\pi}_{11} > 1$ describes an economy inevitably fated to extinction through starvation, and may be typified by the unenviable situation of an invalid on a desert island, who to gain strength to gather one coconut must eat two. It is clear that in such a situation any initial supply of commodity will diminish to zero as time progresses; this situation may then be described as a "starvation economy." The phenomenon which we meet here in its most primitive form can occur also for economies in which there are many commodities; the present paragraph may serve as forewarning of this fact.

Equation (1.11) also shows that ρ is inversely proportional to the amount of capital which must be tied up to produce a unit of the single commodity occurring in our model. Thus equation (1.11) shows that, in this simple case, the rate ρ of profit behaves in an entirely reasonable manner.

4. Critical Discussion of the Model

We have assumed a model in which all relationships are linear and homogeneous, that is, we have assumed "constant returns to scale."

Thus we have neglected all possible economies and diseconomies of scale, and have as well neglected the fact that significant pieces of equipment are not continuously divisible: one cannot build 0.78 of a blast furnace. The force of these objections, however, should be substantially diminished by the prevalence of mass-production, which, by taking systematic advantage of economies of scale, will tend to operate largely in the linear range of production. Many processes may still deviate to a considerable degree from linearity; even so, we remember that calculus tells us that more general functional relationships can often be represented by linear approximations, useful at least over small intervals.

A more substantial difficulty of the above sort arises in those situations in which production makes use of unique natural resources without duplicates; the circumstances in which *economic rents* arise. These resources might be mines, oil wells, agricultural land especially well adapted (as, e.g. by climate) for the production of one or another commodity, etc. These phenomena also are ignored by our model, which implicitly assumes that no producer can have a clear technological advantage over any other producer.

We have also assumed, in writing production coefficients π_{ij} and ϕ_{ij}, that both input and fixed capital costs can be assigned unambiguously to a given output; so that we neglect the common fact that many costs are joint costs rather than costs assignable to a particular output, and that many commodities are produced jointly, some being "by-products" of the production of others. Administrative and auxiliary costs, in particular, would fit only grudgingly into a model like the one before us. Generally speaking, our input-output model portrays the conditions of industrial production rather better than the economic circumstances either of the trade or the service sectors. Since industry amounts only to some 30% of total economic effort, we must be on guard against distortions which may arise from the overly "industrial" view of the economy as a whole which the use of a strict input-output model imposes on us.

We have, of course, neglected all indication of the role of taxes; something will be said on this score later on, however.

Our model is certainly not very explicit as to how transportation costs shall enter, though such costs can be fitted into our model by distinguishing as separate commodities the same physical object in different locations—"ingots f.o.b. mill," and "ingots delivered," and taking "trucking" or "rail transport" as an input required to transform "ingots f.o.b. mill" into "ingots delivered." The same remark might in principle apply to other categories of auxiliary costs, so that one might in principle distinguish between "ingots delivered" and "ingots checked, counted, and recorded," for instance.

A more serious objection is the following. Our model does not take into account the fact that the same commodity can very often be produced by any one of a number of schemes, each requiring the consumption and utilization as fixed capital of different amounts of the "input" commodities, or of the same commodities in differing proportions. On this score, we shall adopt the following point of view. The objective of "efficiency economics" (theory of games, linear programming, and the extensions of these doctrines) is that of determining, according to some given criterion, methods for selecting the "best" production plan, i.e. the plan best according to a criterion of profitability. The efficiency economists usually confine their attention to parts of an economy that are so small that the price structure of the total economy cannot sensibly be affected by the choice of the production scheme within the portion of the economy under study, and under such conditions, the choice of maximum profit under the existing (invariant) prices seems a reasonable one as the criterion for "best." The point of view taken here is that the production coefficients are to be thought of as those obtained by such "local" optimizations, which then becomes the "given" production coefficients in our model. This, of course, neglects the mutual interaction of the "local optimizations" through the price mechanism; nevertheless, we may hope that these neglected interactions cannot affect the general configuration of prices so severely as to introduce grave errors into our more approximate procedure. A more elaborate analysis of this point is reserved for a later lecture (cf. Lecture 17).

In assuming a homogeneous "labor" as input to all sorts of production we are, of course, ignoring the existence of a variety of distinct labor specialties and skills, convertible into each other only with some difficulty, and paid at different rates. We will see subsequently that by generalizing our model to include several labor sectors some of the force of this objection can be removed. A more serious objection along the same lines comes from our inclusion of the real wage bill as the row π_{oj} in the input matrix, as if input to labor was also technologically determined. In the first place, we have ignored the fact that different people will choose to spend their money in different ways. Some of the force of this objection may be met by remarking that the elements π_{oj} may be regarded as statistically average consumption indices. But what is more to the point, the total wage rate is determined not technologically but socially; thus, say, it is presumably possible technologically if not socially to reduce the American wage-rate to wage-rate which supports the urban population of China, i.e., by a factor of twenty. The elements π_{oj} cannot be regarded as technological inputs necessary for the minimal sustenance of labor, but as a socially determined "living wage." Machines will never quarrel with their employers about what inputs they require, but labor may! On the other hand, if the elements π_{oj} are taken to represent "household demand" for the various commodities C_j, their legitimacy as elements of an input-output matrix seems doubtful, since these demands depend upon price, income distribution, etc. We will unravel these difficulties as we go along; one method is to study price theory in an open rather than in a closed model. Neoclassical "Walrasian" economics has had something to say on this score; we shall examine its contributions when we come to study Walras' notion of general equilibrium.

The elements π_{io} are themselves dubious, if not as dubious as the elements π_{oj}. To know these necessary labor inputs we must distinguish the boundary between necessary labor input and "featherbedding"; a boundary which is in continuing dispute. (At the moment these lines are written, a number of railroads are shut down in consequence of a dispute over the question of whether the operation of tugs requires a "float man" in addition to a "deck man.")

A number of criticisms may be directed against the manner in which a uniform rate ρ of profit has been introduced into our model. In the first place, the rate of profit is dependent on the rate of industrial turnover, and will rise, for example, if two shifts rather than one shift are worked with given capital equipment. Thus the number of days for which capital equipment must be tied up in the production of a given output is determined as much socially as technologically. In assuming a rate of profit invariant from industry to industry, and in basing this assumption on an argument involving considerations of long-term equilibrium, we are open to criticism on the grounds that the time required for an economy to come to equilibrium is not short compared with the effect of the perturbing factors, e.g. technological advances which affect the production coefficients π_{ij} and ϕ_{ij}, nor, for that matter, short relative to the apparent frequency of social disturbances and disasters. Nor is the actual economy entirely free of monopoly! But, before involving ourselves with any of these complications, let us study the mathematical features of the model at hand.

LECTURE 2

Basic Mathematics of the Input-Output Model

1. Some Mathematical Notations and Definitions

Capital letters, both Roman and Greek, will be used to denote matrices. Boldface letters will denote column vectors. The individual elements of a matrix A will be denoted by a_{ij}, while the individual elements of a vector $\mathbf{a}$ will be denoted by a_i. The transpose of a matrix will be denoted by a prime, i.e. $B = A'$ means $b_{ij} = a_{ji}$. A row vector will be written as the transpose of a column vector, i.e. $\mathbf{a}'$. The "scalar product" of a row vector $\mathbf{x}'$ by a column vector $\mathbf{y}$ is written $\mathbf{x}' \cdot \mathbf{y}$, by which is meant $\Sigma_i x_i y_i$. This same product may on occasion be written as $\mathbf{y} \cdot \mathbf{x}'$, or, where no confusion can arise, as $\mathbf{x}'\mathbf{y}$ or $\mathbf{y}\mathbf{x}'$. If A is a matrix, the scalar product of $\mathbf{x}'$ with the vector $A\mathbf{y}$ may be written $\mathbf{x}'A\mathbf{y}$, $\mathbf{x}' \cdot A\mathbf{y}$, or $\mathbf{x}'A \cdot \mathbf{y}$. If $\mathbf{a}$ and $\mathbf{b}$ are n-dimensional vectors with components a_i and b_i $(i = 1, \cdots, n)$, then

$\mathbf{a} = \mathbf{b}$ means $a_i = b_i \quad (i = 1, \cdots, n)$;
$\mathbf{a} \geqq \mathbf{b}$ means $a_i \geqq b_i \quad (i = 1, \cdots, n)$;
$\mathbf{a} \geq \mathbf{b}$ means $a_i \geqq b_i \quad (i = 1. \cdots, n)$ and for some component s, $a_s > b_s$;
$\mathbf{a} > \mathbf{b}$ means $a_i > b_i \quad (i = 1, \cdots, n)$.

For matrices A and B, the expressions $A = B$, $A \geqq B$, $A \geq B$, $A > B$, have the similar meanings $a_{ij} = b_{ij}$ (all i,j), $a_{ij} \geq b_{ij}$ (all i,j), $a_{ij} \geq b_{ij}$ (all i,j), and for some r,s $a_{rs} > b_{rs}$.

We note that, using this notation, $\mathbf{a} > \mathbf{b}$ implies $\mathbf{a} \geq \mathbf{b}$ implies

$\mathbf{a} \geqq \mathbf{b}$ so that $\mathbf{a} \geqq \mathbf{b}$ is the weakest statement. Also $\mathbf{a} \leqq \mathbf{b}$ means $\mathbf{b} \geqq \mathbf{a}$, etc.

A vector $\mathbf{a}$ such that $\mathbf{a} > 0$ will be called a *positive vector*, and the set of all n-dimensional positive vectors is called the *positive orthant*. A vector $\mathbf{a}$ such that $\mathbf{a} \geq 0$ will be called a semi-positive vector, and the set of all n-dimensional semi-positive vectors is called the *semi-positive orthant*. A vector $\mathbf{a}$ such that $\mathbf{a} \geqq 0$ will be called a nonnegative vector, and the set of all n-dimensional nonnegative vectors is called the *nonnegative orthant*.

Matrices A such that $A > 0$, $A \geq 0$, and $A \geqq 0$ are called *positive*, *semi-positive* and *nonnegative* matrices respectively.

We will also make frequent use of a number of other standard mathematical terms and notations. The set of all elements (of a sort to be understood from context) having the properties P and Q, and belonging to a set S, will be written

$$\{x \mid x \in S;\ x \text{ has property } P;\ x \text{ has property } Q\}.$$

A set S contained in Euclidean space will be called *closed* if it contains the limit of every convergent sequence of its own points; it will be called *convex* if the three statements $\mathbf{p} \in S$, $\mathbf{q} \in S$, $0 \leq t \leq 1$ imply that $t\mathbf{p} + (l - t)\mathbf{q} \in S$, that is, if S contains the full line-segment between any two of its points. For a discussion of these notions the reader is referred to the work of Karlin cited in the Preface.

2. Theorems on Connected Nonnegative Matrices

Written in terms of vectors, equations (1.6) become

$$\mathbf{p} = \Pi\mathbf{p} + \rho\Phi\mathbf{p} \tag{2.1}$$

where the matrices Π and Φ have the elements π_{ij} and ϕ_{ij} respectively, and have the properties

$$\Pi \geqq 0,\ \Phi \geqq 0 \quad \text{respectively.} \tag{2.2}$$

We intend to show that, under suitable conditions on the matrices Π, Φ, equation (2.1) has a unique solution, $\mathbf{p},\rho$. There is, however, a case in which this assertion is patently false, which must be excluded. Suppose that our economy E divides into two unconnected subeconomies E_1, and E_2, the commodities in E_1 never being used for the production of commodities in E_2, and vice versa. It is then clear that E_1 and E_2 may be totally unrelated, so that E_1 might have a certain natural rate of profit ρ_1, and E_2 a certain natural rate of profit ρ_2 which, by the assertion we wish to make, would both be unique. Then plainly the equation (2.1) for the whole economy E would have no *unique* solution. To rule out this sort of phenomenon, we shall make the plausible hypothesis that every commodity in our economy is ultimately required for the production of every other commodity. More formally, we make the following definitions.

DEFINITION 2.1. *C_j is* directly required *in the production of C_i if $\pi_{ij} > 0$.*

DEFINITION 2.2. *C_j is* ultimately required *in the production of C_i if there is a sequence $C_j, C_{j_1}, C_{j_2}, \cdots, C_{j_l}, C_i$ such that each member of the sequence is directly required in the production of the next member.*

We shall then restrict our study to the case of an economy for which every commodity is ultimately required in the production of every other. Since we have already assumed a homogeneous labor force, the condition would be met if each commodity either ultimately requires labor in its production or is ultimately consumed by laborers as their "input."

Physically separated economies, i.e. several societies with no interactions, could not, of course, be considered as having a homogeneous labor force, and if such a system were considered in a single model like (2.1) the condition would not be met. In this case there would not, in general, be a solution to the system of equations (2.1). In order that such a model have a solution we would have to allow the rate of profit to take on different values for each of the separate economies.

The matrix Π, if it is to describe an economy in which every commodity is ultimately required in the production of every other one, must have the following properties.

(i) $\pi_{ij} \geq 0$.

(ii) For each pair of indices (i,j) there must exist a set of indices $j_1, j_2, \cdots, j_l$ such that

$$\pi_{ij_1}\pi_{j_1j_2}\cdots\pi_{j_lj} > 0. \tag{2.3}$$

DEFINITION 2.3. *A matrix with the properties* (i) *and* (ii) *is called a* connected matrix.

We note in passing that a connected matrix is characterized by the property that no permutation of the columns (or rows) can put the matrix in the form

$$\left[\begin{array}{c|c} X & Z \\ \hline 0 & Y \end{array}\right];$$

here, X and Y are square matrices.

We now turn to the mathematical study of connected matrices. Throughout the following development it is assumed that the matrix A is connected.

THEOREM 2.1. *Let A be a connected nonnegative matrix. There exists a vector $\mathbf{x}$ and a number λ such that $\mathbf{x} \geq 0$ and such that the vector equation*

$$A\mathbf{x} = \lambda\mathbf{x} \tag{2.4}$$

is satisfied. The vector $\mathbf{x}$ is unique to within a multiplicative constant; the number λ is unique and positive. The vector $\mathbf{x}$ has the additional property $\mathbf{x} > 0$.

Such a vector $\mathbf{x}$ is called a positive eigenvector belonging to A, while λ is called the eigenvalue belonging to $\mathbf{x}$.

The proof of our theorem will be broken up into a sequence of lemmas.

LEMMA 2.1 *If $\mathbf{x} \geqq 0$ and $A\mathbf{x} = 0$, then $\mathbf{x} = 0$.*

Proof: $A\mathbf{x} = 0$ plainly implies $A^2\mathbf{x} = 0$ and hence implies $A^l\,\mathbf{x} = 0$ for $l \geq 1$. Thus

$$\sum_i \sum_{j_1} \cdots \sum_{j_l} \sum_k a_{ij_1} a_{j_1 j_2} \cdots a_{j_l k} x_k = 0. \tag{2.5}$$

Since $x_k \geq 0$, each term in this sum is zero. Thus

$$a_{ij_1} a_{j_1 j_2} \cdots a_{j_l k} x_k = 0 \tag{2.6}$$

for all values of the indices $i, j_1, j_2 \cdots j_l$. The connectedness of A requires that one of these coefficients be positive, and therefore $x_k = 0$ for all k. Q.E.D.

LEMMA 2.2. *There exists a positive number λ such that $A\mathbf{x} = \lambda\mathbf{x}$ has a solution with $\mathbf{x} \geq 0$.*

Proof: Pick some fixed $n - 1$ dimensional hyperplane in n-dimensional space intersecting each of the n coordinate axes at a positive value. (The hyperplane

$$\sum_i x_i = 1 \tag{2.7}$$

will do.) Each vector $\mathbf{x} \geq 0$ on the hyperplane (e.g. each solution of (2.7) which lies in the semi-positive orthant) is mapped by A into some point $\mathbf{x}^{(1)}$ in the semi-positive orthant; this follows at once from the fact $A \geq 0$, and from Lemma 2.1. Multiplication by a proper scaling factor depending on $\mathbf{x}^{(1)}$ will send $\mathbf{x}^{(1)}$ into a vector $\mathbf{x}^{(2)}$ lying on the original hyperplane (i.e., take $\sigma = \Sigma_i x_i^{(1)}$, then $x_i^{(2)} = (1/\sigma)x_i^{(1)}$). Put $\mathbf{x}^{(2)} = \phi(\mathbf{x})$. We have then defined, in these two steps, a continuous mapping ϕ of the closed, bounded convex subset

$$\{\mathbf{x} \mid \mathbf{x} \geq 0, \textstyle\sum_{i=1}^n x_i = 1\}$$

of a Euclidean space into itself.

Here we may make use of the famous *fixed point theorem* of Brouwer, whose statement is as follows.

THEOREM. *Let S be a closed bounded convex set in n-dimensional Euclidean space, and let ϕ be an arbitrary continuous mapping of S into itself. Then there exists at least one point $p \in S$ such that $\phi(p) = p$, i.e., at least one point $p \in S$ which remains fixed under the mapping ϕ.*

This is merely the first of many occasions on which we will make use of this remarkable result. A proof may be found in Dunford-Schwartz, *Linear Operators*, Vol. I, p. 468.

Applying Brouwer's theorem, it follows that there exists a vector $\mathbf{x}$ such that $\left(\sum_{i=1}^n x_j^{(1)}\right)\mathbf{x} = A\mathbf{x}$. If we put $\lambda = \sum_{i=1}^n x_i^{(1)}$, it is clear that we have

$$A\mathbf{x} = \lambda\mathbf{x}$$

and

$$\lambda > 0, \mathbf{x} \geq 0. \quad \text{Q.E.D.}$$

LEMMA 2.3. *If $\lambda > 0$, $A\mathbf{x} = \lambda\mathbf{x}$, and $\mathbf{x} \geqq 0$, but $\mathbf{x} > 0$ is false, then $\mathbf{x} = 0$ (i.e. if $\mathbf{x}$ is a nonnegative solution of $A\mathbf{x} = \lambda\mathbf{x}$, and if some component of $\mathbf{x}$ is zero, then all components of $\mathbf{x}$ are zero).*

The proof is similar to that of Lemma 2.1, and will be left to the reader.

Lemmas 2.2 and 2.3 establish the existence of solutions $\mathbf{x}$, λ of the equation $A\mathbf{x} = \lambda\mathbf{x}$ which have the required properties $\mathbf{x} > 0$, $\lambda > 0$. It only remains to establish the uniqueness of this vector and number. Let $B = A'$, i.e. let B denote the transpose of A. Then B is plainly a nonnegative connected matrix. Thus, by what we have already proved, there must exist a vector $\mathbf{y} > 0$ and a number $\mu > 0$ such that

$$B\mathbf{y} = \mu\mathbf{y},$$

i.e., a vector and a number such that

$$\mathbf{y}'A = \mu\mathbf{y}'.$$

Forming the scalar product of each side of the last vector equation with any nonnegative eigenvector $\mathbf{x}$ of A corresponding to the eigenvalue λ, we obtain

$$\lambda\mathbf{y}' \cdot \mathbf{x} = \mathbf{y}'A\mathbf{x} = \mu\mathbf{y}' \cdot \mathbf{x}, \quad \text{i.e.,}$$
$$\lambda\mathbf{y}' \cdot \mathbf{x} = \mu\mathbf{y}' \cdot \mathbf{x}.$$

Thus, since $\mathbf{x} \cdot \mathbf{y}' > 0$, we have

$$\lambda = \mu. \tag{2.8}$$

Now, since $\mathbf{x}$ was any eigenvector of A satisfying $\mathbf{x} \geq 0$, with λ the corresponding eigenvalue, (2.8) implies that all eigenvalues of A be equal to μ, i.e. (2.8) shows that the eigenvalue λ is unique.

Next, we must show that the positive eigenvector $\mathbf{x}$, whose existence we have established, is unique up to a scalar factor. To this end, let $\mathbf{x}^{(1)}$ be an eigenvector, i.e. let

$$A\mathbf{x}^{(1)} = \lambda\mathbf{x}^{(1)}.$$

We define

$$t = \min_j (x_j^{(1)}/x_j),$$

and suppose in addition that the index j for which the indicated ratio assumes its minimum is the index $j = s$. Then plainly

$$\mathbf{x}^{(1)} - t\mathbf{x} \geqq 0$$

while

$$x_s^{(1)} - tx_s = 0.$$

Since

$$A(\mathbf{x}^{(1)} - t\mathbf{x}) = \lambda(\mathbf{x}^{(1)} - t\mathbf{x})$$

it follows by Lemma 2.3 that

$$\mathbf{x}^{(1)} - t\mathbf{x} = 0,$$

thus establishing that the eigenvector is unique to within a multiplicative constant. This completes the proof of Theorem 2.1. Q.E.D.

COROLLARY. *Let B be the transpose of A, and let $\mathbf{x}$ and $\mathbf{y}$ be vectors such that*

(i) $\mathbf{x} \geq 0$ *and* $A\mathbf{x} = \lambda\mathbf{x}$
(ii) $\mathbf{y} \geq 0$ *and* $B\mathbf{y} = \mu\mathbf{y}$.

Then

$$\lambda = \mu.$$

According to Theorem 2.1, the (unique) value λ such that $A\mathbf{x} = \lambda\mathbf{x}$ has a positive solution depends only on A. This number λ will henceforward be called the *dominant of A* and denoted by dom (A). Let us list this as a formal definition.

DEFINITION 2.4. *If A is a connected nonnegative matrix, the unique quantity λ of Theorem 2.1 will be called the* dominant *of A, and written* dom (A), *while the vector $\mathbf{x}$ of Theorem 2.1 (unique up to a positive numerical factor) will be called the* dominant eigenvector *of A.*

We may now reformulate the preceding corollary as follows.

COROLLARY. *Let A be a connected nonnegative matrix, and B be the transpose of A. Then B is a connected nonnegative matrix, and* dom (B) = dom (A).

We now establish some useful facts about the quantity dom (A).

LEMMA 2.4. *Let $k > 0$ be given.*

(i) *If for some vector $\mathbf{z} \geqq 0$ the inequality*

$$A\mathbf{z} \geq k\mathbf{z}$$

is satisfied, then dom $(A) > k$.

(ii) *If for some vector $\mathbf{z} \geqq 0$, the inequality*

$$A\mathbf{z} \leq k\mathbf{z}$$

is satisfied, then dom $(A) < k$.

(i') *If for some vector $\mathbf{z} \geq 0$ the inequality*

$$A\mathbf{z} \geqq k\mathbf{z}$$

is satisfied, then dom $(A) \geq k$.

(ii′) *If for some vector* $\mathbf{z} \geq 0$ *the inequality*

$$A\mathbf{z} \leqq k\mathbf{z}$$

is satisfied, then dom $(A) \leq k$.

Proof of (i): Let B be the transpose of A, and let $\mathbf{y}$ be a positive eigenvector of B; let $\lambda = \text{dom}\,(A) = \text{dom}\,(B)$. Then since every component of $\mathbf{y}$ is positive it follows from the hypothesis

$$A\mathbf{z} \geq k\mathbf{z}$$

that

$$\mathbf{y}' \cdot A\mathbf{z} > k(\mathbf{y}' \cdot \mathbf{z}).$$

Hence

$$\mathbf{z}'B \cdot \mathbf{y} > k(\mathbf{z}' \cdot \mathbf{y})$$

so that

$$\lambda(\mathbf{z}' \cdot \mathbf{y}) > k(\mathbf{z}' \cdot \mathbf{y})$$

and

$$\lambda > k.$$

This proves (i). Statements (ii), (i′), and (ii′) may be proved similarly. Q.E.D.

THEOREM 2.2. dom (A) *is a strictly increasing function of the connected nonnegative matrix* A, *i.e., if* $A^+ \geq A$, *then* dom $(A^+) >$ dom (A).

Proof: Let $A^+ \geq A$.
Then, if $\lambda^+ = \text{dom}\,(A^+)$, there exists a vector $\mathbf{x}^+ > 0$ such that

$$A^+\mathbf{x}^+ = \lambda^+\mathbf{x}^+.$$

Hence

$$A\mathbf{x}^+ \leq \lambda^+\mathbf{x}^+$$

and by Lemma 2.4, $\lambda^+ > \text{dom}\,(A)$. Q.E.D.

The following theorem can be proved in the same way.

THEOREM 2.3. dom (A) *depends continuously on the connected nonnegative matrix* A.

Elaboration of the detailed proof of this theorem is left to the reader.

We are now able to prove the result on prices and rate of profit at which we have aimed.

THEOREM 2.4. *Let* Π *be a connected nonnegative matrix. The equation*

$$\mathbf{p} = \Pi\mathbf{p} + \rho\Phi\mathbf{p} \tag{2.9}$$

has a positive solution $\mathbf{p}$ *with* $\rho \geq 0$ *if and only if* dom $(\Pi) \leq 1$. *In this case* ρ *is unique and* $\mathbf{p}$ *is unique to within a multiplicative constant.*

Proof: If (2.9) has a positive solution $\mathbf{p}$, then by definition, dom $(\Pi + \rho\Phi) = 1$, and conversely. Since, by Theorem 2.2, dom (A) is an increasing function of A, $\rho \geq 0$ implies dom $(\Pi) \leq 1$. Moreover, by Theorem 2.3 and the definition of dom (Φ),

$$\begin{aligned} \text{dom}\,(\Pi + \rho\Phi) &\geq \text{dom}\,(\rho\Phi) \\ &\geq \rho\ \text{dom}\,(\Phi) \end{aligned}$$

i.e. dom $(\Pi + \rho\Phi) \to \infty$ as $\rho \to \infty$. Since dom $(\Pi + \rho\Phi)$ has a value ≤ 1 for $\rho = 0$, tends to ∞ as $\rho \to \infty$, and is a strictly increasing continuous function of ρ, we conclude that dom $(\Pi + \rho\Phi) = 1$ has exactly one solution for $\rho \geq 0$.

On the other hand, if dom $(\Pi) > 1$, then by Theorem 2.2 dom $(\Pi + \rho\Phi) > 1$ and there is no positive solution $\mathbf{p}$ of the equation (2.9). Q.E.D.

COROLLARY. *The quantity* ρ *of the preceding theorem is strictly positive if and only if* dom $(\Pi) < 1$.

The economic interpretation of Theorem 2.4 is immediate. Positive profit is possible only if dom $(\Pi) < 1$. By Lemma 2.4 and the corollary of Theorem 2.1, this condition is equivalent to the condition that there exists a vector $\mathbf{a}' \geq 0$ such that $\mathbf{a}'\Pi \leq \mathbf{a}'$, i.e., to the condition that there exists some scheme of production capable of yielding a physical surplus. The dichotomy dom $(\Pi) < 1$ vs. dom $(\Pi) > 1$ is then evidently a direct generalization of the dichotomy $\bar{\pi}_{\prime\prime} < 1$ vs. $\bar{\pi}_{\prime\prime} > 1$ whose significance was elaborated at the end of Section 3 of the preceding lecture. Only the case dom $(\Pi) < 1$ is of interest; the case dom $(\Pi) > 1$ describes, in the sense explained in the previous lecture, a "starvation economy." If a positive rate of profit is possible, the rate of profit is unique and the prices are positive and unique too within a multiplicative constant. That is, the rate of profit and the ratios of the prices depend only upon the production coefficients π_{ij} and ϕ_{ij}. Moreover, all prices are strictly positive, i.e. nothing is "free."

Thus we are led to conclude that price-ratios are determined by the technological conditions of production; in particular, no considerable role seems to be left for the "supply and demand" considerations which are so central to the customary economic theory of price. This conclusion, of course, is less compelling than it might be owing to the fact that the elements π_{oj} of the input-output matrix are descriptive of consumer preferences. To put our conclusion on a sounder basis, we must study the theory of prices in an open Leontief model. This will be done in the next lecture.

We may add the following interesting argument to what has been said above. Let $q_1, \cdots, q_n$ be the total stocks of commodities $C_1, \cdots, C_n$ available at the beginning of some definite production period. If amounts $a_1, \cdots, a_n$ are to be produced, then **q** and **a** must be related by the inequality

$$q_j \geq \sum_{i=1}^{n} a_i \phi_{ij}, \quad j = 1, \cdots, N.\text{[1]} \tag{2.10}$$

Let a_0 be the smallest amount of labor necessary as input for the production of the commodities $C_1, \cdots, C_N$ in the amounts $a_1, \cdots, a_N$, as determined by the formula

$$a_0 = \sum_{i=0}^{N} a_i \pi_{io}. \tag{2.11}$$

We now inquire as to the conditions under which a proportional increase of at least ρ^* in each stock-total is possible in one production period. During the production period an amount a_j of C_j is produced, but an amount $\Sigma_i a_i \pi_{ij}$ is consumed. The condition that the proportional increase in the level of stocks be at least ρ^* is then evidently

$$a_j - \sum_{i=0}^{n} \pi_{ij} a_i + q_j \geq (1 + \rho^*) q_j.$$

[1] We recall that $\varphi_{oi} = \varphi_{jo} = 0$.

If we use the inequality (2.10) we have

$$a_j - \sum_{i=0}^{N} \pi_{ij} a_i - \rho^* \sum_{i=1}^{N} a_i \phi_{ij} \geq 0, \quad j = 1, \cdots, N.$$

Replacing a_0 by its value as given by (2.11), this may be written as

$$a_j - \sum_{i=1}^{N} (\pi_{ij} + (1 - \pi_{oo})^{-1} \pi_{io} \pi_{oj}) a_i - \rho^* \sum_{i=1}^{N} a_i \phi_{ij} \geq 0, \quad j = 1, \cdots, N.$$

Putting $\pi_{ij} + (1 - \pi_{oo})^{-1} \pi_{io} \pi_{oj} = \tilde{\pi}_{ij}$ and $\tilde{\phi}_{ij} = \phi_{ij}$ for $i,j = 1, \cdots, N$; and writing also $\mathbf{a}'$ for the N-dimensional vector whose components are $a_1, \cdots, a_N$, we may write this last equation as

$$\mathbf{a}' \geqq \mathbf{a}'\tilde{\Pi} + \rho^* \mathbf{a}'\tilde{\Phi}.$$

It follows by Lemma 2.4 and the corollary of Theorem 2.1 that dom $(\tilde{\Pi} + \rho^* \tilde{\Phi}) \leq 1$. Now, we saw in the preceding lecture that the quantity ρ was determined by the condition that the equation

$$\mathbf{p}' = \tilde{\Pi}\mathbf{p}' + \rho\tilde{\Phi}\mathbf{p}'$$

have a positive vector solution $\mathbf{p}'$. Thus, by Definition 2.4, ρ is determined by the condition that dom $(\tilde{\Pi} + \rho\tilde{\Phi}) = 1$. It follows at once by Theorem 2.2 that our hypothetical rate ρ^* of stock increase satisfies the inequality

$$\rho^* \leq \rho.$$

Heuristic interpretation of this inequality yields a statement to which economics has been so devoted that it may be called the Main Theorem of Economics: no planned scheme of production can (without lowering wages, which is painful, or improving efficiency, which is doubtful) yield a greater annual increase in stocks than the normal rate of profit of free competition. More flamboyantly, free enterprise is the best of all possible systems! Our subsequent analysis will not fail to find flies in this ointment. Nevertheless, our assertion has a decidedly nontrivial force.

LECTURE 3

Theory of Prices in the Open Leontief Model. Some Statistics

1. Determination of Prices by ρ

We were only able to extend the open system (1.4) of equations for prices to the closed system (1.6) by making use of the coefficients π_{oj} representing the components of the real wage bill, and writing the additional equation

$$p_0 = \sum_{j=0}^{n} \pi_{oj} p_j. \tag{3.1}$$

Since, as has already been pointed out, these particular coefficients are hardly determined as much by technology as are the other production coefficients (including the coefficients π_{jo} which denote the amount of labor required for the production of a unit of C_j) it is of interest to study the theory of prices in a corresponding model in which equation (3.1) is suppressed, i.e., to study the theory of prices in an open Leontief model.

Suppressing equation (3.1), we are left with the homogeneous equations

$$p_i = \pi_{io} p_o + \sum_{j=1}^{n} \pi_{ij} p_j + \rho \sum_{j=1}^{n} \phi_{ij} p_j, \quad i = 1, \cdots, n. \tag{3.2}$$

This is a set of n equations for the set of $n + 1$ unknowns consisting of ρ and the n ratios of the prices $p_0, p_1, \cdots, p_n$. We may consequently surmise that these equations determine all but one of the

$n + 1$ unknowns. To see that this is in fact the case, let us first pass to an inhomogeneous system by introducing the variables $\tilde{p}_1 = p_1/p_0$, $\tilde{p}_2 = p_n/p_0, \cdots, \tilde{p}_n = p_n/p_0$. In terms of these variables we may write

$$(3.3) \qquad \tilde{p}_i - \sum_{j=1}^{n} \pi_{ij}\tilde{p}_j - \rho \sum_{j=1}^{n} \phi_{ij}\tilde{p}_j = \pi_{io}, \quad i = 1, \cdots, n.$$

This is a nonhomogeneous set of n equations in the $n + 1$ unknowns $\tilde{p}_1, \cdots, \tilde{p}_n$ and ρ. Guided by the expectation that the specification of any one of the unknowns will make the solution of the system determinate, we shall begin by studying this system as a system of equations determining the unknowns $\tilde{p}_1, \cdots, \tilde{p}_n$ in terms of the parameter ρ.

Let $\mathring{\Pi}$ be the matrix whose elements are π_{ij} $(i,j = 1, \cdots, n)$, i.e. $\mathring{\Pi}$ is the matrix obtained by deleting the zero row and the zero column from Π. We shall assume in the rest of the present lecture, for convenience' sake and in order to be able to make ready reference to the results of the preceding lecture, that $\mathring{\Pi}$ is connected. It would not be hard to generalize our analysis to cover cases in which this assumption is not satisfied. Let $\mathring{\Phi}$ be similarly defined as the matrix whose elements are ϕ_{ij} $(i,j = 1, \cdots, n)$, i.e. $\mathring{\Phi}$ is the matrix obtained by deleting the zero row and the zero column from Φ. Let $\mathbf{v}$ be the vector whose components v_i are π_{io}. The vector whose elements are $\tilde{p}_i$ $(i = 1, \cdots, n)$ we shall denote simply by $\tilde{\mathbf{p}}$. Equations (3.3) then become

$$\tilde{\mathbf{p}} - \mathring{\Pi}\tilde{\mathbf{p}} - \rho\mathring{\Phi}\tilde{\mathbf{p}} = \mathbf{v}$$

or

$$(3.4) \qquad (I - \mathring{\Pi} - \rho\mathring{\Phi})\tilde{\mathbf{p}} = \mathbf{v}.$$

We now ask under what circumstances the matrix $(I - \mathring{\Pi} - \rho\mathring{\Phi})$ has an inverse for, when the inverse exists, we obtain the explicit solution

$$(3.5) \qquad \tilde{\mathbf{p}}(\rho) = (I - \mathring{\Pi} - \rho\mathring{\Phi})^{-1}\mathbf{v}$$

for the prices $\tilde{p}_i$ as a function of the parameter ρ. (We note again that no assumption as to the detailed nature of consumer demand enters into our considerations.) A suitable answer to this question may be obtained from the following theorem.

THEOREM. *Let A be an arbitrary matrix. Let μ (possibly complex) denote any number for which there exists a nonzero solution $\mathbf{z}$ (again possibly complex) to*

$$A\mathbf{z} = \mu\mathbf{z}.$$

(Even in the complex case, such a number is called an eigenvalue of the matrix A.) If every such μ satisfies $|\mu| < 1$, then the sum $\sum_{k=0}^{\infty} A^k$ converges and is equal to the inverse of $I - A$; that is

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

This theorem is the matrix-theoretic generalization of the scalar formula:

$$1/(1 - x) = \sum_{k=0}^{\infty} x^k \quad \text{for} \quad x < 1.$$

Proof of this theorem would involve us in an unprofitably extensive discussion of the theory of matrices. The reader interested in examining such a proof should consult Dunford-Schwartz, *Linear Operators*, Vol. 1, p. 559.

We now prove a number of lemmas which permit us to apply the preceding theorem to the problem at hand.

LEMMA 3.1. *If M is a connected nonnegative matrix, and $M\mathbf{z} = \mu\mathbf{z}$ for some nonzero vector $\mathbf{z}$, then $|\mu| \leq$ dom (M).*

Proof: Let

$$M\mathbf{z} = \mu\mathbf{z}.$$

Then

$$\left|\sum_j m_{ij}z_j\right| = |\mu|\,|z_i|$$

so that

$$\sum_j m_{ij}\,|z_j| \geq |\mu|\,|z_i|.$$

Let $y_j = |z_j|$ so that $\mathbf{y}$ is a real nonnegative vector. Then

$$\sum_j m_{ij}y_j \geq |\mu|\,y_i.$$

Hence, by Lemma 2.4, dom $(M) \geq |\mu|$. Q.E.D.

We omit the very similar proof of the following slightly stronger statement.

COROLLARY 3.2. *Under the hypotheses of Lemma* 3.1 *we have either*

$$\mu = \text{dom}\,(M)$$

or

$$|\mu| < \text{dom}\,(M).$$

LEMMA 3.3. *If M is a connected nonnegative matrix, and if* dom $(M) < 1$, *then the inverse $(I - M)^{-1}$ exists and is the sum of the series $\sum_{k=0}^{\infty} M^k$.*

Proof: This follows at once from Lemma 3.1 and the theorem which precedes it. Q.E.D.

LEMMA 3.4. *If M is a nonnegative matrix, so are all its powers.*

Proof: Immediate by induction. Q.E.D.

LEMMA 3.5. *If M is a connected nonnegative matrix, and* dom $(M) < 1$, *the inverse $(I - M)^{-1}$ is a nonnegative matrix.*

Proof: This follows at once from Lemmas 3.3 and 3.4. Q.E.D.

LEMMA 3.6. *If M and M' are nonnegative matrices, M being connected, while* dom $(M) < 1$, dom $(M') < 1$, *and $M \leq M'$, it follows that*

$$(I - M)^{-1} < (I - M')^{-1}.$$

Proof: It is plain that the connectedness of M and the relationship $M \leq M'$ implies the connectedness of M'. We may show by induction that $M^k \leqq (M')^k$; thus it follows from Lemma 3.3 that $(I - M)^{-1} \leqq (I - M')^{-1}$. On the other hand, let i and i' be any two indices of the matrix M. Since $M \leq M'$, there is (by definition of the relationship) at least one element $m_{jj'}$ of M which is *strictly smaller* than the corresponding element $m'_{jj'}$ of M'. Since M is connected, there exists a sequence $i, i_1, \cdots, i_l, j$ and a sequence $i_{l+1}, \cdots, i_q, i'$ such that the product

$$m_{ii_1}m_{i_1i_2}\cdots m_{i_lj}m_{j'i_{l+1}}\cdots m_{i_qi'}$$

is nonzero. Hence

$$(3.6)\quad m_{ii_1}m_{i_1i_2}\cdots m_{i_lj}m_{jj'}m_{j'i_{l+1}}\cdots m_{i_qi'} < m'_{ii_1}m'_{i_1i_2} \cdots m'_{i_lj}m'_{jj'}m'_{j'j_{l+1}}\cdots m'_{i_qi}.$$

It follows at once from this inequality, and from the inequality $m_{st} \leq m'_{st}$ valid for all indices s and t, that the i,i' element of the matrix M^{q+3} is strictly smaller than the corresponding element of the matrix $(M')^{q+3}$. Our conclusion now follows immediately from the formulae for $(I - M)^{-1}$ and $(I - M')^{-1}$ given by Lemma 3.3. Q.E.D.

COROLLARY. *If M is a connected nonnegative matrix, and $M \leq M'$, M' is also a connected nonnegative matrix.*

It follows from Lemma 3.3 that a sufficient condition that $I - \mathring{\Pi} - \rho\mathring{\Phi}$ have an inverse, is the condition dom $(\mathring{\Pi} + \rho\mathring{\Phi}) < 1$. Let $\rho_{\max}$ be the least upper bound of the set of ρ such that dom $(\mathring{\Pi} + \rho\mathring{\Phi}) < 1$. Since by Theorems 2.2 and 2.4 dom $(\mathring{\Pi} + \rho\mathring{\Phi})$ is a strictly increasing continuous function of ρ, it follows at once that $\rho_{\max}$ may equivalently be defined as the unique solution of the equation dom $(\mathring{\Pi} + \rho_{\max}\mathring{\Phi}) = 1$. That is, $\rho_{\max}$ is the unique number for which there exists a positive solution $\mathbf{x}_{\max}$ to the system

$$(3.7)\qquad (\mathring{\Pi} + \rho_{\max}\mathring{\Phi})\mathbf{x}_{\max} = \mathbf{x}_{\max}.$$

This system may be written

$$(3.8)\quad (\mathbf{x}_{\max})_i = \sum_{j=1}^{n} \pi_{ij}(\mathbf{x}_{\max})_j + \rho_{\max}\sum_{j=1}^{n} \phi_{ij}(\mathbf{x}_{\max})_j, \quad i = 1, \cdots, n.$$

Comparing the system (3.8) with the system (3.2), we see that $\rho_{\max}$ is the rate of profit which obtains when p_0, the wage rate, is zero, and that $\mathbf{x}_{\max}$ is the corresponding vector of commodity prices. By Theorem 3.2 and its corollary, $\rho_{\max} > 0$ if and only if dom $(\mathring{\Pi}) < 1$, i.e., if and only if, with the wage rate at zero, the physical machine of production is capable of producing a surplus. (Compare the corresponding discussion in the preceding lecture; since an economy violating this condition is not worth considering, we will make the assumption dom $(\mathring{\Pi}) < 1$ in all that follows.)

Clearly, $\rho_{\max}$ is the rate of profit which is obtained by taking $\mathbf{v} = 0$ in (3.3), i.e., the rate of profit which would obtain if no wages at all are paid. Then for $\rho < \rho_{\max}$, i.e., for rates of profit such that positive wages are paid, dom $(\mathring{\Pi} + \rho\mathring{\Phi}) < 1$, and $(I - \mathring{\Pi} - \rho\mathring{\Phi})$ has an inverse. Thus, referring once more to the fundamental equation (3.4), determining prices, we now conclude that not only are the prices $\tilde{\mathbf{p}}(\rho)$ uniquely determined independently of demand once a rate of profit $0 < \rho < \rho_{\max}$ is specified, but also, using Lemma 3.6, that the prices so determined are, as they must be, strictly positive.

Lemma 3.6, applied to the solution (3.5) of the system (3.2), yields the following theorem immediately.

THEOREM 3.7. *All components of $\tilde{\mathbf{p}}(\rho)$ are strictly increasing with ρ for $0 \leq \rho < \rho_{\max}$.*

Thus, as ρ increases, the relative price of every single commodity, as measured by the hourly wage rate, increases. It follows that not only ρ but in fact any one of the price-ratios $\tilde{p}_i$ may be taken as a parameter determining a unique solution to the system (3.3). If we know the price of any commodity in terms of the hourly wage rate, it follows that ρ, and hence all the price ratios, are determined by our system of equations (3.2).

We emphasize once more for the benefit of the reader familiar with the conventional "neoclassical" or "marginal utility" analysis that, considering the almost vanishing role played by consumer preference in the above analysis, we have before us very strong presumptive evidence against the marginal utility theory (or more precisely, against its special significance). More detailed comparison of the input-output and marginal utility theory will be made in subsequent lectures.

2. Generalization to Several Labor Sectors

All our arguments up to the present point were based upon the assumption of a single, homogeneous, labor sector. We shall now consider briefly the modifications which would arise if we took the inhomogeneity of the labor force into account, introducing several kinds of labor as commodities C_0, C_{-1}, C_{-2}, etc., with corresponding production coefficients $\pi_{i,0}, \pi_{i,-1}, \pi_{i,-2}$, where $\pi_{i,-k}$ is the amount of labor of type k required for the production of one unit of C_i. Let the ratios of the wages paid to the different kinds of labor be $1, x_{-1}, x_{-2}, \cdots$. Then equations (3.3) become

$$\tilde{p}_i - \sum_{j=1}^{n} \pi_{ij}\tilde{p}_j - \rho \sum_{j=1}^{n} \phi_{ij}\tilde{p}_j = \pi_{io} + \sum_{k} \pi_{i,-k}x_{-k} \tag{3.9}$$

which shows that by a simple substitution for π_{io} of

$$\pi_{io} + \sum_{k} \pi_{i,-k}x_{-k} \tag{3.10}$$

the preceding analysis may be brought to include the case in which there is a nonhomogeneous labor force for which the ratios of the wages are known. The prices are then determined by ρ and by the ratios of the wage levels for each of the several kinds of labor.

3. Relation of the Leontief Model with Standard Economic Statistics

To relate the present model to statistics taken from one or another actual national economy, we must exhibit the definition, in terms of the present model, of the various conventional statistical headings in terms of which a national economy is ordinarily described. The quantities whose significance appears most immediately are the totals $a_1, \cdots, a_n$ of production, in a given period, of the commodities $C_1, \cdots, C_n$ of the economy. Such totals will ordinarily be available as series titled "annual production of coke" or "annual generation of electric power," etc. The vector $\mathbf{a}'$ whose components are $a_1, \cdots, a_n$ may be called the *total national product vector* (for the given period) *in real terms*. It is a list of the total amounts of all commodities produced, irrespective of whether or not the commodity is subsequently consumed in the production of further commodities.

Even here, where our concept is apparently clear-cut, and hence to a yet more significant extent in dealing with less straightforward theoretical notions, we run up against annoying problems of practical and statistical definition. Is "total production of vegetables" to include products grown for immediate consumption in farm gardens? We might wish that it did: but the statistician may legitimately object that to gather reliable figures on such production is extremely difficult. Hence one settles for data describing vegetable production as it appears on the commercial market, thereby becoming statistically accessible. This problem, which may appear mild when one thinks of material commodities, becomes pressing when we pass to the consideration of services. In the first place, let us remark that unless services are included among the commodities $C_1, \cdots, C_n$, our list $a_1, \cdots, a_n$ of total production figures may constitute an entirely inadequate description of economic activity. But to include services raises statistical problems. Consider, as an exceptionally definite service category, "haircuts." Should totals for this service include only the activity of the professional barber, or should it include the widespread efforts of amateurs as well? What then about "shaves," commonly a self-performed service, or "preparation of cooked food" in which the restauranteur stands side-by-side with a larger group of housewives? Or consider the situation to be dealt with in treating a varied service category like medical care. It hardly be either feasible or instructive to present a complete breakdown of this general activity into its components; one rather expects a summary total heading. To obtain such a heading, one has hardly any choice but to weight individual terms according to prices actually charged. This is made all the more necessary by the fact that in attempting to compile useful general totals of economic activity, which may with some legitimacy be called by such imposing titles as *national income, gross national product,* etc., the statistician sorely needs the check which is provided by definitions which make "total valuation of goods and services produced" equal to "total income expended" upon these same goods and services. But this means valuing things at their actual prices, even though, for instance, this may lead, implicitly if not explicitly, to the "services" provided by an extortionist appearing in a national income account as if these services were justly describable as "extortions, x in number, to the total value of y dollars." The proper treatment of taxes, of course, provides similar problems.

Another serious problem is the treatment of house rents. On the expenditure side, these sum to a significant total; on the product side, we are somewhat perplexed to find the hypothetical commodity "shelter" altogether immaterial in form, its "production" following immediately from the presence of certain fixed capital equipment. In order that the purchase of housing for personal use should not lead to excessively large subtractions from such a heading as "gross national product," it is then necessary to include an item covering imputed rental income of householders, as little as such a practice seems to conform to the assumption implicit in our input-output model.

Even the assignment of production or expenditure to a given period may be questionable, as significant items may be "carried over" or "carried forward" in statistics as well as on corporate books. And how, for instance, shall additions to inventory and

inventories be measured, amidst constantly changing price levels and technology? We find, as usual in science, that the clear-cut distinctions of a theoretical model apply to the complex empirical situation only approximately. With all these reservations in mind, we may continue to describe the significance, in terms of our model, of the main headings of national income accounting.

The components of the vector $\mathbf{a}' - \mathbf{a}'\mathring{\Pi}$ are the total amounts of the various commodities produced net of commercial, agricultural, and industrial consumption; i.e., the total amounts of commodities produced but not subsequently consumed in the production of other commodities. This then is the total bill of goods available as income for individual consumption, increase of stocks or fixed capital investment, national defense purposes, etc. To estimate the size of this vector by a single number it is appropriate to multiply each production figure by the unit price of the commodity involved and to sum. Taking actual prices in this calculation, we arrive at the total commonly called *net national product* (NNP). It is appropriate, however, to make a correction for the distortion of prices by sales, property, and excise taxes, and various indirect business liabilities of this sort. Without such a correction, an increase in the sales tax rate or the property tax rate appears as an increase of apparent product. The correction is ordinarily made by subtracting the total of business indirect tax liability from the NNP (and by making a small additional correction for the effects of corporate pension plans and a few other items listed as "business transfer payments"); this yields the total commonly called *national income* (NI). It is then the heading national income which most closely approximates our theoretical expression $(\mathbf{a}' - \mathbf{a}'\mathring{\Pi}) \cdot \tilde{\mathbf{p}}(\rho)$ (cf. (3.3) and the following paragraph). The given definition of net national product has the advantage that it is equal to the sum of personal consumption expenses (including property and sales tax payments), net investment, net exports, and government purchases of goods and services; this equality provides a useful statistical check.

If we add to the net national product the total estimated depreciation of fixed capital for the period under consideration, we obtain the *gross national product.* This somewhat hybrid heading gains significance from the fact that it is hard to make a firm distinction between capital depreciation in the sense of actual loss of usefulness through wear and capital depreciation as investment in plant modernization. Wear of fixed capital should be included, in accordance with our model, on a suitable prorated basis, in the elements of the circulating capital matrix π_{ij} and subtracted as is indicated by the formula $\mathbf{a}' - \mathbf{a}'\mathring{\Pi}$; equipment depreciated through obsolescence and to clear the way for modernization might perhaps be regarded as a form of final output, even though it may not be a very useful form of output.

The difference between national income and the total wage bill, deflated by the average rate of hourly wages, is given in our model by the formula

$$\mathbf{a}'(I - \mathring{\Pi}) \cdot \tilde{\mathbf{p}}(\rho) - \mathbf{a}' \cdot \mathbf{v}. \tag{3.11}$$

(Compare (3.3) and the paragraph following for definition of these quantities.) This is a measure of the total value of commodities remaining for investment, special luxury consumption, military purposes, or other use, after wages are paid (and before saving out of wages). It may be called the *national dividend*, and may, of course, be stated either in deflated dollar or current-dollar terms.

The ratio of the national dividend to the national income is called the *retention fraction*. Explicitly, the retention fraction $r(\rho)$ is given by

$$r(\rho) = [\mathbf{a}' \cdot (I - \mathring{\Pi})\tilde{\mathbf{p}}(\rho) - \mathbf{a}' \cdot \mathbf{v}]/[\mathbf{a}' \cdot (I - \mathring{\Pi})\tilde{\mathbf{p}}(\rho)]. \tag{3.12}$$

In order to make the order of magnitude of the various quantities described in the present section apparent, we now present some statistical information.

Our figures are based on data given by the Department of Commerce, modified by the following rough procedures so as to conform more closely to our theoretical headings:

(a) Proprietors' income is separated into a "salary" and a "profit" part, the separation being based on the assumption that unincorporated business is approximately as profitable as incorporated business.

(b) Debits are then made for personal income tax and for indirect tax payments, the stated totals for each of these taxes being debited on a proportional basis against each liable form of income.

(c) The relevant headings in the resultant partition of Gross National Product are then summed to give an estimate of National Dividend.

TABLE Ia

U. S. 1957 Approximate Salaries Before and After Taxes, Showing Estimated Deductions for Income, Excise, and Property Taxes (Billions of Dollars)

	Est. before taxes	Est. personal tax correction	Est. indirect tax correction	Est. after taxes
Wages and salaries of employees	247	32	28	187
Net pension, social security, unemployment insurance, and other public and private social transfer payments	15	0	2	13
Farm proprietors' income	12	2	1	9
Professional and business self-employment income	21	3	3	15
Rental income of persons	12	1	1	10
Total wages and salaries	307	38	35	234

TABLE Ib

U. S. 1957 Approximate Profits and Interest Before and After Personal and Indirect Taxes, Showing Estimated Deductions for Taxes (Billions of Dollars) [a]

	Est. before taxes	Est. personal tax correction	Est. indirect tax correction	Est. after taxes
*Noncorporate profits	10	1	1	8
*Corporate dividends	12	2	1	9
*Corporate retained profits	9	0	0	9
*Net private interest	13	2	1	10
*Government interest payments	6	0	0	6
Total profits and interest	50	5	3	42

[a] Items marked * to be included in national dividend.

TABLE Ic

U. S. 1957 Government Expenditures on Goods and Services
(Billions of Dollars) [a]

*State and local government net investment	4
Additional state and local government expenditures	32
Federal nonmilitary expenditures	6
*Federal military expenditures	44
Total government expenditures on goods and services	86

[a] Items marked * to be included in national dividend

TABLE Id

Relation of Tables Ia–c to Gross National Product Account
(Billions of Dollars)

Total wages and salaries after taxes	234
Private dividends and interest after taxes	42
Total government expenditures on goods and services	86
Government surplus and foreign transfers	3
Subtotal (NI estimate)	365
Actual national income	367
Depreciation	38
Excise, property, and miscellaneous indirect taxes	38
Total (GNP estimate)	441
Actual Gross National Product	440

TABLE Ie

U. S. 1957 Estimated National Dividend
(Billions of Dollars)

Private profits and interest (Table Ib)	42
Federal military expenditures	44
Net state and local government investment	4
National Dividend	90

From the approximate figures in the tables, we may estimate the retention fraction as $\frac{90}{365}$, or 25%.

The rate of profit ρ, as it would appear in our input-output model, is the ratio of national dividend to total capital plant value.

We take the statistics in Tables IIa and IIb from a study of capital wealth by Lampman (*Review of Economics and Statistics*, (1959)); c.f. also the extensive study by Goldsmith: *A Study of Saving in the United States*, 3 Vols., Princeton University Press (1956).

Using Tables Ie and IIb, we may estimate the rate ρ of profit, as it would appear in our model, as the quotient of national dividend

TABLE IIa

Approximate Capital Account of United States, 1953
(Billions of Dollars) (after Lampman)

Residences	295	Individual property or use
Consumer durables	123	
Public improvements	100	
Commercial buildings and structures	160	
Industrial equipment	135	
Industrial and trade inventories	107	
Gold and silver	27	
Total	947	
Land valuations	209	
	1156	

TABLE IIb

Estimated U. S. 1957 Productive Capital Account
(Billions of Dollars)

Commercial buildings and structures	160
Industrial equipment	135
Inventory	107
Public improvements	100
Total productive capital acct.	502

over total productive capital, thus $\frac{90}{502}$ or 18%. We may illustrate the significance of this figure by stating that the productive capital of the United States could be used to duplicate itself in something more than five years.

This profit rate ρ which describes the physical economy is to be reconciled with the rather different average rate of yield on stocks and bonds as follows. Somewhat more than $\frac{1}{4}$ of the national dividend of 90 billion never appears on corporate books as profit at all, appearing instead as withholding tax from wages and salaries; another $\frac{1}{4}$ or so does not appear as disposable profit because of corporate taxes. These two factors would alone reduce the apparent profit rate to something like 8.5%. About $\frac{1}{4}$ of the remaining profit is retained by corporations for internal expansion further reducing the apparent profit rate to 6.5%. Finally, the practice of the Federal and State governments of imposing taxes and paying part of these taxes as interest to government bond holders, coupled with the circumstance that the institution of land rent means that land-values enter into the computation of profit rates, means that instead of dividing by the total value 502 billion of capital improvements, we must rather divide by the sum of private capital improvements, government bond values, and commercial land values, approximately $400 + 300 + 150 = 850$ rather than 500. This larger denominator would reduce the apparent profit rate to approximately 3.8%, which is to be compared to the current (1960) average bond yield of 3.5% on tax-free municipal bonds, 4.8% on taxable corporate bonds, and average stock yield of 3.6% on corporate stocks.

4. A Rough Statistical Test

The model used here for the study of prices is based on the assumption of constant return to scale, or, in other words, it is based on the assumption that the π_{ij} and ϕ_{ij} are constants and not functions of the total amounts a_i of production of the various commodities. The fundamental result indicated by our study of the model is that the relative prices depend only upon the production coefficients (if labor is included as a commodity) or, more cautiously (when labor is not included) that the relative prices depend only upon the production coefficients and the rate of profit ρ. From this we must conclude that if we assume a constant ρ, the relative prices should be independent of the production level. We may use the fluctuation of production levels over the course of the business cycle to obtain a rough empirical check on this theoretical conclusion. Over periods of time short enough to consider the Π and Φ constant the present

theory then indicates that the relative prices should be much the same in a boom period and in a period of recession. Let us compare this conclusion with the empirical data on the business cycle as precisely elucidated by W. C. Mitchell.

TABLE IIIa

Average Peak and Trough Production Levels for 4 Cycles 1919–1938 (after Mitchell)

	Peak	Trough	Variation, %
Industrial production	120	87	33
Auto production	130	70	60
Cotton	120	90	30
Housing contracts	130	90	40
Factory pay	125	85	40

TABLE IIIb

Average Relative Prices for the 1919–1938 Peak and Trough Production Level Years (after Mitchell) Deflated by Wholesale Price Level

	Peak	Trough	Variation, %
Wholesale prices of finished goods	100	100	0
Wholesale prices for semi-manufactured goods	104	97	7
Raw materials	105	96	9
Wholesale foods	100	98	2
Retail foods	101	97	4
Pig iron	106	94	12
Farm prices	106	96	10

Table IIIa shows the production levels of various key U. S. industries averaged over four peak years and over four trough years in the period 1919–1938, indicating production level fluctuations of 30% to 60%. Table IIIb then shows the prices of various commodities (adjusted to the wholesale prices of finished goods) averaged over the same four peak years and over the same four trough years. The differences in prices are certainly much smaller than the differences in production levels, perhaps small enough to indicate a certain relevance of the present model, and to justify its further study.

Our model suggests that the larger relative variations in raw materials prices (as compared to prices of manufactured goods) ought to be associated with a smaller validity of the assumption of constant returns to scale. We will cite some evidence supporting this suggestion, as it applies in the particular case of hide production, in a subsequent lecture. Of course, many other circumstances give rise to price variations over the course of the business cycle. Speculation, that is, adjustment to anticipated future conditions rather than present conditions, must be taken into account; as well as differences in the storability of goods of various types. The phases of the business cycle are not static equilibrium situations.

LECTURE 4

Concluding Discussion of the Leontief Model

1. Some Comparative Statistics

It is of interest to compare the United States gross national product account presented in the last lecture with the corresponding account for other national economies.

Of course, all international comparisons of income accounts are dubious to a certain extent. Thus, for instance, definitions of wages and of salaries vary; price ratios and foreign exchange rates vary, so that dollar equivalents are hard to establish; the extent to which a given economy is properly represented by commercial data making no allowance for subsistence economy varies also.

The British account for 1957 is as shown in Table IV below.

From the approximate figures below, we may estimate the retention fraction r as 3.1/17.0, or 18%, compared to the U. S. figure of 25%.

We shall not attempt to make an estimate of the capital resources of Britain, and hence will not calculate the rate of return on productive capital for Britain.

Let us note that the input-output model must be applied even more cautiously to Britain than to the U. S., since the British import-export business amounts to 4 billion pounds sterling or over 20% of the value of Britain's gross national product, while the corresponding figure for the U. S. is some 20 billion dollars or less than 5% of the value of the gross national product.

We may compare the two above sets of figures with data for an

TABLE IV

Britain 1957 Approximate Gross National Product Account, in Billions of Pounds Sterling [a,b]

Wages and salaries of employees	9.6
Pension, social security, unemployment benefits, and national health	1.6
Proprietors self-employment income	1.2
Total wages and salaries after all taxes	12.4
*Rent, dividends, and interest	1.3
*Retained public and private corporate net profit	0.2
Total private sector income after all taxes	13.9
*Military expenditure	1.6
Nonmilitary central government expenditure, goods and services	0.4
Local governmental expenditure	1.1
Total real national income	17.0
Government surplus	0.6
National income	17.6
Excises less subsidies	2.5
Net national product	20.1
Capital consumption allowance	1.8
Gross national product	21.9

[a] Based on data from the *National Income and Expenditure* volume of the Central Statistical Office of Great Britain, figures reduced to U. S. Department of Commerce basis.

[b] Items marked * are to be included in national dividend.

ambitious underdeveloped country by examining the national accounts of China; here, of course, we are limited by the paucity of reliable data. The following figures are taken from the careful analysis by Li: *Industrial Development of Communist China*. The account of production is shown first in Table V. From Tables V and VI we may estimate the retention fraction r as $\frac{23}{110}$, or 21%.

Estimates for the gross product of the U. S. S. R. are also rather sparse. We give, in Table VII, estimates prepared by H. Block, and published in *Trends in Economic Growth. A Comparison of the Western Powers and the Soviet Bloc. A Study Prepared for the Joint Committee on the Economic Report by the Legislative Reference Service of the Library of Congress*, United States Government Printing Office, Washington, 1955.

The retention fraction r here is at the exceptionally high level of

TABLE V

China 1957 Estimated National Income Account (Billions of Yuan) [a]

Agriculture	45
Industry and mining	24
Construction	5
Transport	4
Trade	14
Service	18
Total national income	110

[a] The official exchange rate is 1 Yuan = \$0.43 U. S.

TABLE VI

China 1957 Estimated Net Investment and National Dividend Account (Billions of Yuan) [a]

*Producers' equipment	3
*Commercial and government construction	4
Housing construction	1
*Construction for agriculture, flood control, and forestry	2
*Commercial and industrial inventory investment	0.5
*Agricultural inventory investment	1
*Net investment outside state plan	1
*Additional miscellaneous investment and military procurement	11.5
Total national dividend	23.0

[a] Items marked * to be included in national dividend.

TABLE VII

Estimated U. S. S. R. 1953 Gross National Product, Corrected for Excise Taxes (after H. Block) [a]

Consumption	51
Social services	9
Administration	4
Defense	18
Net investment	25
National income	107
Depreciation	3
Gross national product	110

[a] Prices of physical components estimated in billions of U. S. 1953 Dollars.

40%, compared to 18% for the United Kingdom, 22% for China, and 25% for the U. S.

2. The Leontief Model and Labor Productivity

The retention fraction is not an indicator of the standard of living, as is evident from the fact that the retention fractions for the U. S., Britain, and China, are quite comparable, but the standards of living are quite different. The standard of living depends also, and more significantly, upon the values of the production coefficients, and in particular upon the coefficients π_{jo} describing the amount of labor required for the production of various commodities, i.e., upon "labor productivity." Table VIII below allows a rough comparison of the amount of labor required for the production of some typical basic commodities in Britain and in the United States, thus explaining the higher standard of living in the U. S. Of course, factors such as quality of product, hours of labor expended, etc., are neglected in our crudely quantitative comparison, but it is hardly likely that they could affect the comparison to any considerable degree.

TABLE VIII

Rough Estimates of Comparative Labor Productivity in the U. S. and the United Kingdom, 1957 [a]

	U. S.		United Kingdom		Approx. ratio to U. S.
	Millions employed	Production	Millions employed	Production	
Vehicles	2		1.2		3
mill. cars		5.5		0.9	
mill. trucks		1.2		0.3	
Textiles, bill. yds.	1	12	0.9	2.5	5
Coal mining, mill. tons	0.25	500	0.75	220	7
Agriculture, forestry, and fishing	6		1		2
mill. doz. eggs		5		1	
mill. acres of grain		150		7	
mill. tons potatoes		24		6	
mill. tons fish		2.5		29	

[a] The data for this table are from the *Europa Yearbook*, 1960.

TABLE IX

Ratios of Labor Productivity in Selected Industries, U. S./Britain (1948)

Tin Containers	4.96	Grain milling	1.86
Cardboard containers	4.24	Bicycles	1.88
Pig iron	4.91	Rubber tires	2.03
Wool yarn	4.53	Jute yarn	1.77
Radio receiving tubes	3.74	Beet sugar	0.85
Cigarettes	3.63	Building bricks	1.77
Wool carpets and rugs	3.28	Paint brushes	1.74
Glass containers	3.06	Cotton piece goods	1.78
Soap	2.89	Boots and shoes	1.67
Paper sacks	2.77	Rope and twine	1.61
Watches	2.59	Margarine	1.23
Ice cream	2.14	Cement	1.39
Animal feeds	2.06	Razor blades	1.12
Biscuits	2.18	Canned fish	1.20
Malt liquors	1.98	Ice	0.69

TABLE X

Rough Comparative Estimates of Agricultural Labor Productivity for various Countries, Mid-1950's [a]

	Total pop. (mill.)	Employed (mill.)	Agriculture employed (mill.)	Production (mill. tons)		Ratio to U. S.
				Grain	Potatoes	
U. S.	180	60	8	150	12	1
Great Britain	50	24	1	7	6	2
W. Germany	52	24	4	12½	26	2
Czechoslovakia	13	6	2	4½	9	3
Poland	29	17	8	12	35	4
U. S. S. R.	210	105	45	141	86	4
Japan	90	43	15	15	10	12
Turkey	24	14	9½	14	1½ (rice)	12
Mexico	33	10	6	6	—	20
China	600	300	260	175	23	27
Indo-China	12	7	5	—	3½ (rice)	28
U. S. (1910)	—	—	—	—	—	3

[a] The data for this table is from the *Europa Yearbook*, 1960.

More precise ratios of labor productivity in the year 1948 have been given by M. Frankel, *American Economic Review* (1955), v. 45, p. 94. We have reproduced some of Frankel's data in Table IX above.

Table X gives crude estimates of labor productivity in agriculture (computed on a grain and potatoes basis) for a number of economies. The close relation between this one parameter and the general socio-economic state of a nation is striking.

3. Some Power Series Expansions

Since according to Theorem 3.7 the price vector $\tilde{\mathbf{p}}(\rho)$ is a strictly increasing function of ρ, it follows from the definition (3.12) of the retention fraction $r(\rho)$ that the retention fraction is a strictly increasing function of ρ.[1] It is plain from the definition (3.12) that $r(\rho)$ has the value zero for $\rho = 0$, in which case the national dividend is zero, and has the value one for $\rho = \rho_{\max}$, which case, as we saw in our preceding lecture, is defined by the hypothetical circumstance that the wage rate sinks to zero. Thus we may regard ρ as a function of r. Figure 1 illustrates the relationship between ρ and r.

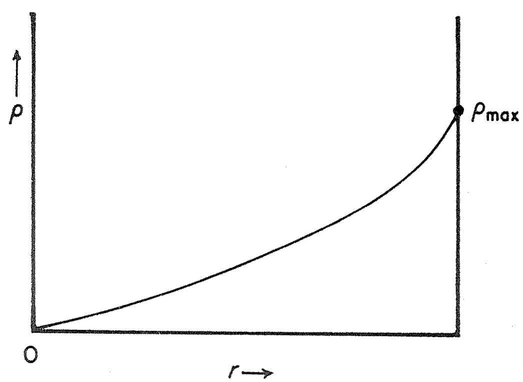

Fig. 1. Retention fraction and rate of profit.

[1] The monotonicity of this relationship corresponds to the popular adage "you can't have your cake and eat it too." This rueful fact apparently constitutes a dialectical contradiction. I quote from Li: "The inverse relationship between the rate of accumulation and the rate of consumption is considered as one of the 'internal contradictions' to be resolved." See Mao Tse-Tung "Concerning the Correct Disposition of the Problem of the People's Internal Contradictions," *Jen-min jih-pao*, June 19, 1957.

It is interesting to express the rate of profit ρ as a function of r by evaluating the coefficients in the expansion:

$$\rho(\gamma) = \sum_{k=1}^{\infty} \gamma_k r^k, \tag{4.1}$$

making use of (3.11) and (3.12), i.e. by solving (3.12) for ρ, obtaining a solution in the form of a power series expansion. To do this let us first define

$$f(\rho) = \mathbf{a}'(I - \mathring{\Pi})\tilde{\mathbf{p}}(\rho) \tag{4.2}$$

so that, from (3.12), it follows that

$$r = (f(\rho) - f(o))/f(\rho)$$

and hence

$$f(\rho) = f(o)/(1 - r).$$

Thus

$$f(\rho) = f(o)(1 + r + r^2 + \cdots)$$

i.e.

$$f(\rho) - f(o) = f(o) \sum_{k=1}^{\infty} r^k$$

or

$$(\rho f_\rho(o) + \tfrac{1}{2}\rho^2 f_{\rho\rho}(o) + \cdots) = f(o) \sum_{k=1}^{\infty} r^k$$

which may be written as

$$\left(f_\rho(o) \sum_{k=1}^{\infty} \gamma_k r^k + \tfrac{1}{2} f_{\rho\rho}(o) \left(\sum_{k=1}^{\infty} \gamma_k r^k\right)^2 + \cdots\right) = f(o) \sum_{k=1}^{\infty} r^k. \tag{4.3}$$

The coefficients γ_k may now be evaluated by setting equal the coefficients of like powers of r from both sides of the equation; we obtain the equations

$$f_\rho(o)\gamma_1 = f(o) \tag{4.4a}$$

$$f_\rho(o)\gamma_2 + \tfrac{1}{2}\gamma_1^2 f_{\rho\rho}(o) = f(o) \quad \text{etc.} \tag{4.4b}$$

Thus, to first order in r,

$$\rho = (f(o)/f_\rho(o))\, r + 0(r^2) \tag{4.5}$$

To evaluate the coefficients $f(o)$, $f_\rho(o)$, etc. in terms of the production coefficients, thereby obtaining a more explicit expression for the ratio $f(o)/f_\rho(o)$, we use equation (3.5):

$$\tilde{\mathbf{p}}(\rho) = (I - \mathring{\Pi} - \rho\mathring{\Phi})^{-1}\mathbf{v}.$$

This formula must now be expanded as a power series in ρ. The desired expansion may be carried out by a procedure which is quite standard in the theory of matrices. First, we use the formula $A^{-1} = B^{-1}(AB^{-1})^{-1}$ to separate the factor $(I - \mathring{\Pi})^{-1}$ out of the expression $(I - \mathring{\Pi} - \rho\mathring{\Phi})^{-1}$. This gives

$$\tilde{\mathbf{p}}(\rho) = (I - \mathring{\Pi})^{-1}(I - \rho\mathring{\Phi}(I - \mathring{\Pi}))^{-1}\mathbf{v}.$$

Next we expand the expression $(I - \rho\mathring{\Phi}(I - \mathring{\Pi})^{-1})^{-1}$ in a geometric series, making use of the theorem whose statement follows formula (3.5) (which theorem is surely applicable for sufficiently small ρ). This gives

$$\tilde{\mathbf{p}}(\rho) = (I - \mathring{\Pi})^{-1}\mathbf{v} + \rho(I - \mathring{\Pi})^{-1}\mathring{\Phi}(I - \mathring{\Pi})^{-1}\mathbf{v} + \rho^2(I - \mathring{\Pi})^{-1}\mathring{\Phi}(I - \mathring{\Pi})^{-1}\mathring{\Phi}(I - \mathring{\Pi})^{-1}\mathbf{v} + \cdots$$

which gives an explicit power-series expansion for the price-vector in terms of ρ. Thus

$$(\partial/\partial\rho)\,\tilde{\mathbf{p}}(o) = (I - \mathring{\Pi})^{-1}\mathring{\Phi}(I - \mathring{\Pi})^{-1}\mathbf{v}$$

or

(4.6) $$(\partial/\partial\rho)\,\ddot{\mathbf{p}}(o) = (I - \mathring{\Pi})^{-1}\mathring{\Phi}\tilde{\mathbf{p}}(o);$$

from (4.1), we then obtain

$$f_\rho(o) = \mathbf{a}'(I - \mathring{\Pi})(\partial/\partial\rho)\,\tilde{\mathbf{p}}(o) = \mathbf{a}'\mathring{\Phi}\tilde{\mathbf{p}}(o).$$

We may consequently write

(4.7) $$f(o)/f_\rho(o) = [\mathbf{a}'(I - \mathring{\Pi})\tilde{\mathbf{p}}(o)/\mathbf{a}'\mathring{\Phi}\tilde{\mathbf{p}}(o)].$$

Hence, finally, an expression for ρ as a function of r valid to second order in r is given by

(4.8) $$\rho(r) = [\mathbf{a}'(I - \mathring{\Pi})\tilde{\mathbf{p}}(o)/\mathbf{a}'\mathring{\Phi}\tilde{\mathbf{p}}(o)]r + 0(r^2.)$$

In this connection we note also the equation

(4.9) $$\tilde{\mathbf{p}}(\rho) = (I - \mathring{\Pi})^{-1}\mathbf{v} + 0(\rho)$$

valid to first order.

Equations (4.8) and (4.9) together express two well-known forms of the classical "labor theory" of prices and profit.

The first-order equation (4.8) expresses a theory of prices with the following heuristic significance: let the "labor content" of a commodity be defined as the actual labor consumed in its production, plus the labor content of all the commodities consumed in its production. Letting L_i denote the labor content of a unit of C_i, the following equation is an immediate consequence of the definition:

(4.10) $$L_i = \pi_{io} + \sum_{j=1}^{n} \pi_{ij}L_j = v_i + \sum_{j=1}^{n} \pi_{ij}L_j.$$

Thus $(I - \mathring{\Pi})\mathbf{L} = \mathbf{v}$. Comparing this equation with (4.9), we see at once that $\mathbf{L} = \tilde{\mathbf{p}}(o)$. Thus, taking $\tilde{\mathbf{p}}(\rho) = \mathbf{L}$ corresponds to expanding about $\rho = 0$ (i.e., $r = 0$) and omitting all higher terms. Similarly, the second order equation (4.8) for ρ is that which is obtained by expanding the function $\rho(r)$ about $r = 0$, and by subsequent omission of all terms of higher order than the first.

4. Excise Taxes

It is quite easy to apply the price theory which has been developed to the situation in which excise taxes are imposed on some or all of the commodities in our model economy. Let τ_i, $1 \leq i \leq n$, be the excise-rate on commodity C_i (figured as a percentage of basic price). Then the reasoning which led us to the equation (1.5) leads us, if we take these taxes into account, to the equation

(4.11) $$p_i = \pi_{io}p_o + \sum_{j=1}^{n} \pi_{ij}(1 + \tau_j)p_j + \rho\sum_{j=1}^{n} \phi_{ij}(1 + \tau_j)p_j.$$

Denoting p_i/p_o by $\tilde{p}_i$ and setting $v_i = \pi_{io}$ as before, we have

(4.12) $$\tilde{p}_i - \sum_{j=1}^{n} \pi_{ij}(1 + \tau_j)\tilde{p}_j - \rho\sum_{j=1}^{n} \phi_{ij}(1 + \tau_j)\tilde{p}_j = \mathbf{v}_i.$$

If we write $\mathring{\Pi}_\tau$ and $\mathring{\Phi}_\tau$ for the matrices whose elements are $\pi_{ij}(1 + \tau_j)$ and $\phi_{ij}(1 + \tau_j)$ respectively, we may write the equations (4.12) in vector form as

$$(4.13)\qquad (I - \mathring{\Pi}_\tau - \rho\mathring{\Phi}_\tau)\tilde{\mathbf{p}}(\rho,\tau) = \mathbf{v},$$

with the solution

$$(4.14)\qquad \tilde{\mathbf{p}}(\rho,\tau) = (I - \mathring{\Pi}_\tau - \rho\mathring{\Phi}_\tau)^{-1}\mathbf{v}.$$

If the tax rates τ are small, we may use the techniques of the preceding section to make a power series expansion of the price-vector $\tilde{\mathbf{p}}(\rho,\tau)$ in the quantities τ_i. To first order in the tax rates τ_i we would have

$$(4.15)\qquad \tilde{\mathbf{p}}(\tau,\rho) = (I - \mathring{\Pi} - \rho\mathring{\Phi})^{-1}\mathbf{v} + (I - \mathring{\Pi} - \rho\mathring{\Phi})^{-1}(\hat{\Pi}_\tau + \hat{\Phi}_\tau)(I - \mathring{\Pi} - \rho\mathring{\Phi})^{-1}\mathbf{v} + 0(\tau^2),$$

where $\hat{\Pi}_\tau$ and $\hat{\Phi}_\tau$ denote the matrices whose elements are $\pi_{ij}\,\tau_j$ and $\phi_{ij}\tau_j$ respectively. Equation (4.15) may be written in terms of the vector $\tilde{\mathbf{p}}(\tau)$ of prices before the imposition of taxes as

$$(4.16)\qquad \tilde{\mathbf{p}}(\tau,\rho) = \tilde{\mathbf{p}}(\rho) + (I - \mathring{\Pi} - \rho\mathring{\Phi})^{-1}(\hat{\Pi}_\tau + \hat{\Phi}_\tau)\tilde{\mathbf{p}}(\rho).$$

The question which is traditional in the theory of excise taxes, to wit, the question "to where are excise taxes shifted" is thus answered by our theory (cf. especially (4.13)) as follows. Excise taxes are shifted unrestrictedly through the economy, finally resting upon the individual consumer, unless these taxes are absorbed in whole or part by a fall in the profit rate ρ. Does ρ fall? About this question the theory tells us nothing.

In the theory of excise taxes our input-output model then gives us the same ready if somewhat approximate answer that it gives us in the simple theory of prices.

5. Concluding Remarks

Our discussion of the theory of prices in the Leontief input-output model may now be (at least temporarily) concluded. As we have seen, this price theory is simple, definite in its point of view, and unambiguous in its conclusions. Since our model omits so many of the factors affecting prices in the real economy, however, we would be ill-advised to take this theory as more than an elaboration of important principles. We may take the prime significance of our discussion of the Leontief model to lie less in the price-theory which has been elaborated than in the circumstance that the model provides a reasonably realistic framework within which the material functioning of a real economy may be described. Writing an input-output matrix gives us a bird's eye view of the overall flow of commodities in an economy, i.e. the process whereby the growth of certain stocks proceeds via subtraction from other stocks. We shall now make use of this framework to give a theory, like that of Metzler, of the business cycle.