PART II

THEORY OF CONSUMER BEHAVIOR AND DEMAND

CHAPTER 5

MARSHALL'S ANALYSIS OF DEMAND

Initially Alfred Marshall initially worked with objective demand curves. However by working backwards, he developed its psychological basis by 1890. While it would be interesting to trace out this development, it will be bypassed; rather we are concerned with the final product. To deal with Marshall's analysis of demand, we first consider his characterization of wants in relation to activities; second his characterization of utility; third his derivation of the demand curve from utility; and lastly his discussion of the price elasticity of demand, Giffen good, and other characteristics of the demand curve. [work on]

Wants in Relation to Activities

Marshall assumed that an individual's wants or desires for economic goods are given, independent of the social life and activities of the individuals and hence are irreducible datum for analysis. He further argued that the satisfaction of given wants were independent of the social activities designed to fulfill these wants. As a result, it was now possible to define *rationality* unambiguously as any activities that are directed towards fulfilling these wants and *economic efficiency* as the activities that maximize the possible wants-satisfactions given factor endowments, technology, and wants.[1] This normative use of economic concepts is peculiarly dependent on the fixity of wants and rationality. On the one hand, the satisfaction of known wants supplies the only possible norm in terms of which the desirability or efficiency of an economic process can be judged. If the ends themselves come to vary as a function of the process of their attainment, the standard no longer exists. On the other hand, the same process of want-satisfaction is itself the most general and obvious meaning of rationality of action. The

[1] It should be noted that under perfect information, rationality and optimal economic efficiency merge into one.

very concept of rationality is meaningless without reference to given ends while non-rational want-satisfaction is non-sense except in terms of divergence from a rational type.

In contradistinction to the position that wants are given and independent from the activities that fulfill them, Marshall also argued that wants may adjust to the activities pursued to fulfill them. He felt that the individual had two kinds of wants:

1. there are the wants of the savage that are determined by biology and the need to survive, that is they are natural. Hence the activities that the savage pursued to fulfill their wants do not change them.
2. there are the wants of the 'civilized man':

> …through the brute and the savage alike have their preferences for choice morsels, neither of them cares much for variety for its own sake. As, however, man rises in civilization, as his mind becomes developed, and even his animal passions begin to associate themselves with mental activities, his wants become rapidly more subtle and more various; and in the minor details of life he begins to desire change for the sake of change, long before he has consciously escaped from the yoke of custom. [Marshall 1972: 73]

Hence, Marshall, argued, the activities that are undertaken to satisfy civilized or social wants creates new wants. Thus civilized wants result from social activities:

> …although it is man's wants in the earliest stages of his development that give rise to his activities, yet afterwards each new step upwards is to be regarded as the development of new activities giving rise to new wants, rather than of new wants giving rise to new activities. [Marshall 1972: 76]

Consequently individuals in society pursue activities as an end in themselves. However, if Marshall wanted to use the terms rationality, maximization, and economic efficiency, wants must be given independently of the social activities. Hence he argued that the higher study of civilized wants must come after the main body of economic analysis had been developed which, in turn, must be confined to the elementary natural wants. Thus Marshall assumed for his analysis of demand that wants are given and independent of the activities pursued to fulfill these wants. [Marshall 1972: 76–7]

Utility

Law of Diminishing Marginal Utility

Maintaining his previous position, Marshall argued that the use-values of an economic good is determined in the eyes of the consumer. Denoting use-value as utility and stating that the consumption of economic goods satisfy consumer wants, he argued that utility is correlative to want and the total utility (or total satisfaction of the want) increases with increased consumption of the economic good, but at a decreasing rate. This was due, Marshall argued, to human nature—that is the *law of diminishing marginal utility* is grounded in the human psyche and thus can be said to be natural. He then connects the exchange value of an economic good to its marginal utility by stating that the amount of good a consumer is just induced to purchase is called his/her *marginal purchase* because the consumer is on the margin of doubt whether it is worth his while to buy it. Thus the utility the consumer gets from his marginal purchase is called *marginal utility*. In this manner the marginal purchase (or the demand price) is the measure of utility and hence an indirect measure of wants.

Marshall noted that the law of diminishing marginal utility, hence his entire argument on the relationship between wants and marginal purchase, depends on wants being fixed. Hence it is no exception to the law that the more good music a person hears, the stronger his/her taste for it likely to become since the observations range over some period of time and the person's tastes are not the same at the beginning as at the end. Thus for the law to hold true, time must not *move* or in other words economic analysis must take place at a point in time.[2]

So far we have dealt with a consumer's utility with respect to a single good. However, a consumer consumes more than a single good or, in other words, has a multiplicity of wants that are satisfied by many different goods. Thus the consumer's total utility is a function not only of a single good, but of all goods consumed. In mathematical notation, the consumer's *utility function* is delineated as:

$U = \mu(y_1, y_2, \ldots, y_n)$

where y_i is the absolute amount of the ith economic good.

The above utility function is in a very general form; its specific form depends on the assumptions made concerning wants. If wants are assumed to be independent from each other and their utilities additive (which implies that utilities are homogeneous), then the utility function takes on the following form:

$U = \mu_1(y_1) + \ldots + \mu_n(y_n)$.

Such a function is called an *additive utility function* (or additive and separable utility function). The implication of this utility function is that a consumer's total utility can be obtained by adding together the utilities obtained from each of the economic goods consumed. However, such a utility function was not acceptable to Marshall since he believed that the utilities derived from various goods were interdependent (see Marshall 1972: 109). But his derivation of the demand curve is only consistent with a additive utility function; hence we shall work with it.

The Mathematics of Utility

1. Utility of a single good Y_i:

let y_i be the quantity of Y_i consumed and the total utility of good Y_i be $U_i = \mu_i(y_i)$

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[2] Briefly relate to the Stigler and Becker arguments (1978).

a. for all $y_i > 0$, then $U_i = \mu_i(y_i) > 0$.

b. $dU_i/dy_i = d\mu_i(y_i)/dy_i > 0$ where $d\mu_i(y_i)/dy_i$ is the marginal utility of Y_i.

This says that the total utility increases with greater consumption of Y_i.

c. $d^2U_i/dy_i^2 = d\mu_i^2(y_i)/dy_i^2 < 0$ which is the rate of increase in total utility decreases.

2. Utility of Goods, Y_i where $i = 1, 2, \ldots, n$

a. for all $y_i > 0$, the consumer's total utility will be $U = U_1 + U_2 + \ldots + U_n$ or

$U = \mu_1(y_1) + \mu_2(y_2) + \ldots + \mu_n(y_n) > 0$.

b. $\partial U/\partial y_i = \partial \mu_i(y_i)/\partial y_i > 0$ which means that the increase in total utility depends solely on Y_i because of the assumed separability of wants.

c. $\partial^2U/\partial y_i^2 = \partial^2\mu_i(y_i)/\partial y_i^2 < 0$ which means that marginal utility is diminishing.

d. $\partial^2U/\partial y_i\partial y_j = \partial^2\mu_i(y_i)/\partial y_i\partial y_j = 0$ because of the separability of wants (further discussed in ch. 6).

Demand Curve for a n-Good Utility Function

To develop a demand curve, Marshall considered a utility function with *n* goods but dealt with one good:

> Now let us translate this law of diminishing [marginal] utility into terms of price. Let us take an illustration from the case of a commodity such as tea, which is in constant demand and which can be purchased in small quantities. Suppose, for instance, that tea of a certain quality is to be had at 2s. per lb. A person might be willing to give 10s. for a single pound once a year rather than go without it altogether; while if he could have any amount of it for nothing he would perhaps not care to use more than 30 lbs. in the year. But as it is, he buys perhaps 10 lbs. in the

> year; that is to say, the difference between the satisfaction which he gets from buying 9 lbs. and 10 lbs. is enough for him to be willing to pay 2s. for it: while the fact that he does not buy an eleventh pound, shows that he does not think that it would be worth an extra 2s. to him. That is, 2s. a pound measures the utility to him of the tea which lies at the margin or terminus or end of his purchases; it measures the marginal utility to him. If the price which he is just willing to pay for any pound be called his *demand price*, then 2s. is his *marginal demand price*. And our law may be worded: the larger the amount of a thing that a person has the less, other things being equal (i.e. the purchasing power of money, and the amount of money at his command being equal), will be the price which he will pay for a little more of it: or in other words his marginal demand price for it diminishes….At one and the same time, a person's material resources being unchanged, the marginal utility of money to him is a fixed quantity, so that the prices he is just willing to pay for two [or *n*] commodities are to one another in the same ratio as the utility of those two [*n*] commodities. [Marshall 1972: 79–80]

To formalize the argument, we can do the following (this needs to be redone):

a. Givens: money income – M; wants (tastes); and purchasing power of money, that is the prices of the *n*-goods.

b. the individual's n-good utility function: $U = \mu_1(y_1) + \ldots + \mu_n(y_n)$

c. budget constraint: $M = p_1y_1 + \ldots + p_ny_n$

   the consumer is assumed to spend his entire income to obtain the greatest amount of utility as possible; this constraint is fulfilled as long as

   $\partial U/\partial y_i > 0$, which, as shown above, is one of the properties of the utility function.

d. Let us now consider the proposition that the individual wants to maximize his total utility subject to his budget constraint:

$\max L = \mu_1(y_1) + \ldots + \mu_n(y_n) + \lambda(M - p_1y_1 - \ldots - p_ny_n)$

This is called a *Lagrangian function* and λ is a *Langrange multiplier*. To find the first order conditions (FOC) for a local maximum we differentiate L with respect to y_i and λ to obtain

$L_1 = \partial\mu(y_1)/\partial y_1 - \lambda p_1 = MU_1 - \lambda p_1 = 0$

……………………………………….

$L_n = \partial\mu(y_n)/\partial y_n - \lambda p_n = MU_n - \lambda p_n = 0$

$L_\lambda = M - p_1y_1 - \ldots - p_ny_n = 0$

(i) assuming that the individual is maximizing its utility, it will distribute his/her income over the *n* goods to the point that will equalize the ration MU_i/p_i among all *n* goods and the marginal utility of money λ: $\lambda = MU_1/p_1 = \ldots = MU_n/p_n$.

(ii) the above relationship can also be written as $\lambda = MU_i/MU_j = p_i/p_j$.

To determine whether a local maximum actually exists, we have to look at the second order conditions in which each L_i is differentiated with respect to y_i, i = 1, …, n and λ:

$L_{11} = \partial^2\mu(y_1)/\partial y_1^2$; $L_{12} = \partial^2\mu(y_1)/\partial y_1\partial y_2 = 0$; … ; $L_{1n} = \partial^2\mu(y_1)/\partial y_1\partial y_n = 0$; $L_{1\lambda} = -p_1$

………………………………………….

$L_{n1} = \partial^2\mu(y_n)/ \partial y_n\partial y_1^2 = 0$; $L_{n2} = \partial^2\mu(y_1)/\partial y_n\partial y_2 = 0$; … ; $L_{nn} = \partial^2\mu(y_n)/\partial y_n^2$; $L_{n\lambda} = -p_n$

$L_{\lambda 1} = -p_1$; ………………………………………………………… ; $L_{\lambda n} = -p_n$; $L_{\lambda\lambda} = 0$

Putting this into a *bordered hessian matrix*, we have

$$\begin{bmatrix} \partial^2\mu(y_1)/\partial y_1^2 & 0 \ldots\ldots\ldots\ldots & 0 & -p_1 \\ 0 & \partial^2\mu(y_2)/\partial y_2^2 & 0 & -p_2 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots \\ 0 & 0 \ldots\ldots\ldots & \partial^2\mu(y_n)/\partial y_n^2 & -p_n \\ -p_1 & -p_2 \ldots\ldots\ldots\ldots & -p_n & 0 \end{bmatrix}$$

The sufficient condition for maximization is if all the bordered-preserving principal minors of the hessian matrix of the order k have the sign $(-1)^k$, $k = 2, \ldots, n$. Hence, by simple inspection we can see that

$$D = \begin{vmatrix} \partial^2\mu(y_1)/\partial y_1^2 & 0 \ldots\ldots\ldots & 0 & -p_1 \\ 0 & \partial^2\mu(y_2)/\partial y_2^2 & & -p_2 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots \\ 0 & 0 \ldots\ldots\ldots & \partial^2\mu(y_n)/\partial y_n^2 & -p_n \\ -p_1 & -p_2 \ldots\ldots\ldots\ldots & -p_n & 0 \end{vmatrix} > 0$$

Now we are in the position to show that the individual's demand curve always slopes downward. Starting with the first order conditions, the *demand function* for the ith good is derived by solving the first order conditions for a utility maximization equilibrium:

$y_1^e = f_1(p_1, \ldots, p_n, M)$

……………………..

$y_n^e = f_n(p_1, \ldots, p_n, M)$

$\lambda^e = f_\lambda(p_1, \ldots, p_n, M)$

Now substituting y_i^e and λ^e into the first order conditions we

$\partial\mu(y_1^e)/\partial y_1 - \lambda^e p_1 \equiv 0$

……………………………………….

$\partial\mu(y_n^e)/\partial y_n - \lambda e p_n \equiv 0$

$M - p_1 y_1^e - \ldots - p_n y_n^e \equiv 0$

Now differentiating with respect to p_i, we get

$$[\partial^2\mu(y_1^e)/\partial y_1^2][\partial y_1^e/\partial p_i] - [\partial\lambda^e/\partial p_i]p_1 \equiv 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$[\partial^2\mu(y_i^e)/\partial y_i^2][\partial y_i^e/\partial p_i] - [\partial\lambda^e/\partial p_i]p_i - \lambda^e \equiv 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$[\partial^2\mu(y_n^e)/\partial y_n^2][\partial y_n^e/\partial p_i] - [\partial\lambda^e/\partial p_i]p_n \equiv 0$$

$$- p_1\partial y_1^e/\partial p_i - \ldots - p_i\partial y_i^e/\partial p_i - y_i^e - \ldots - p_n\partial y_n^e/\partial p_i \equiv 0$$

Putting it in the appropriate form, we have

$$\begin{bmatrix} \partial^2\mu(y_1)/\partial y_1^2 & 0 & \ldots\ldots & 0 & -p_1 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ 0 & \partial^2\mu(y_i)/\partial y_i^2 & & 0 & -p_i \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ 0 & 0 & \ldots\ldots & \partial^2\mu(y_n)/\partial y_n^2 & -p_n \\ -p_1 & \ldots\ldots -p_i & \ldots\ldots & -p_n & 0 \end{bmatrix} \begin{bmatrix} \partial y_1^e/\partial p_i \\ \ldots\ldots \\ \partial y_i^e/\partial p_i \\ \ldots\ldots \\ \partial y_n^e/\partial p_i \\ \partial\lambda^e/\partial p_i \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \ldots \\ \lambda^e \\ . \\ 0 \\ y_i^e \end{bmatrix}$$

Solving for $\partial y_i^e/\partial p_y$ using Cramer's Rule, we have $\partial y_i^e/\partial p_y = \underline{y^e D_{in+1} + \lambda^e D_{ii}}/D < 0$.

However, instead of simply applying Cramer's Rule to solve for $\partial y_i^e/\partial p_y$, let us instead take a closer look at λ^e and $\partial\lambda^e/\partial p_i$. It has already been stated that λ^e is the marginal utility of money.[3] Thus, dealing with $\partial\lambda^e/\partial p_i$ can be described as the change in the marginal utility of money with respect to a change in the price p_i. This is quite close to Marshall's notion of the change in the marginal utility of money:

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[3] This can be shown in the following manner. Substituting y_i^e into the utility function, we have $U = \mu_1(y_1^e) + \ldots + \mu_n(y_n^e)$. Now differentiating with respect to M, we get $\partial U/\partial M = [\partial\mu_1(y_1^e)/\partial y_1][\partial y_1^e/\partial M] + \ldots + [\partial\mu_n(y_n^e)/\partial y_n][\partial y_n^e/\partial M]$ or $\partial U/\partial M = MU_1[\partial y_1^e/\partial M] + \ldots + MU_n[\partial y_n^e/\partial M]$. From the first order conditions we know that $MU_i = \lambda^e p_i$. Thus by substituting we get $\partial U/\partial M = \lambda^e p_1\partial y_1^e/\partial M + \ldots + \lambda^e p_n\partial y_n^e/\partial M$ or $\partial U/\partial M = \lambda^e[p_1\partial y_1^e/\partial M + \ldots + p_n\partial y_n^e/\partial M]$. Now when the budget constraint is differentiated with respect to M we get: $M - p_1 y_1^e - \ldots - p_n y_n^e \equiv 0 \rightarrow \partial U/\partial M \equiv p_1\partial y_1^e/\partial M + \ldots + p_n\partial y_n^e/\partial M \equiv 1$. Substituting this result into the above, we find that $\partial U/\partial M = \lambda^e$ or λ^e is the marginal utility of money.

> The substance of our argument would not be affected if we took account of the fact that, the more a person spends on anything the less power he retains of purchasing more of it or of other things, and the greater is the value of money to him (in the technical language every fresh expenditure increases the marginal value of money to him.) But though its substance would not be altered, its form would be made more intricate without any corresponding gain; for there are very few practical problems, in which the corrections to be made under this head would be of any importance. [Marshall, 1972, p. 109]

In short, Marshall assumed that $\partial\lambda^e/\partial p_i = 0$ or λ^e is fixed or given. As we shall see, by assuming $\partial\lambda^e/\partial p_i = 0$, Marshall has in effect assumed away the income effect.

Returning to the above discussion but keeping in mind that the marginal utility of money is fixed, the above results can be rewritten as the following

$$[\partial^2\mu(y_1^e)/\partial y_1^2][\partial y_1^e/\partial p_i] \equiv 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$[\partial^2\mu(y_i^e)/\partial y_i^2][\partial y_i^e/\partial p_i] - \lambda^e \equiv 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$
$$[\partial^2\mu(y_n^e)/\partial y_n^2][\partial y_n^e/\partial p_i] \equiv 0$$
$$-p_1\partial y_1^e/\partial p_i - \ldots - p_i\partial y_i^e/\partial p_i - y_i^e - \ldots - p_n\partial y_n^e/\partial p_i \equiv 0$$

Now since $\partial^2\mu(y_j^e)/\partial y_j^2 \neq 0$, then $\partial y_j^e/\partial p_i = 0$ where $j \neq i$. Therefore the above system of equations reduces to:

$$[\partial^2\mu(y_i^e)/\partial y_i^2][\partial y_i^e/\partial p_i] \equiv \lambda^e$$

$$-p_i\partial y_i^e/\partial p_i \equiv y_i^e$$

Now solving for $\partial y_i^e/\partial p_i$ the slope of the demand curve for the ith good (and noting that only the first equation needs to be considered), we have:

$$\partial y_i^e/\partial p_i = [\lambda^e]/[\partial^2\mu(y_i^e)/\partial y_i^2] < 0 \text{ since } \partial^2\mu(y_i^e)/\partial y_i^2 < 0 \text{ and } \lambda^e \text{ is constant .}$$

Thus the demand curve for the ith good slopes downward because of the law of diminishing marginal utility.

To recapitulate, the demand curve is derived from the utility maximization equilibrium as represented by the first order conditions. Given those equilibrium positions, which are denoted as demand functions, the price of the ith good is varied to see how the quantity demanded reacts. It is shown that as p_i decreases the quantity of Y_i would increase. Therefore for given different prices for Y_i, there exists different equilibrium quantities demand of Y_i with lower prices being associated with higher equilibrium quantities demanded due to the law of diminishing marginal utility.

Market Demand Curve

General Law of Demand

Marshall obtained the market demand curve by summing horizontally the demand curves of each individual for good Y_i. Hence Marshall proclaimed that there was one general law of demand:

> The greater the amount to be sold, the smaller must be the price which it is offered in order that it may find purchasers; or, in other words, the amount demanded increases with a fall in price, and diminishes with a rise in price…. The price will measure the marginal utility of the commodity to each purchaser individually…. [Marshall 1972: 84]

By tying the market demand curve to the utility function of the individual consumers, Marshall developed a psychologically based demand curve. Marshall also noted that the demand curve was valid for only a given moment in economic time and given conditions, such as tastes, prices of rival goods, and money income. If any of these conditions changed then the demand curve would be altered.

Price Elasticity of Demand

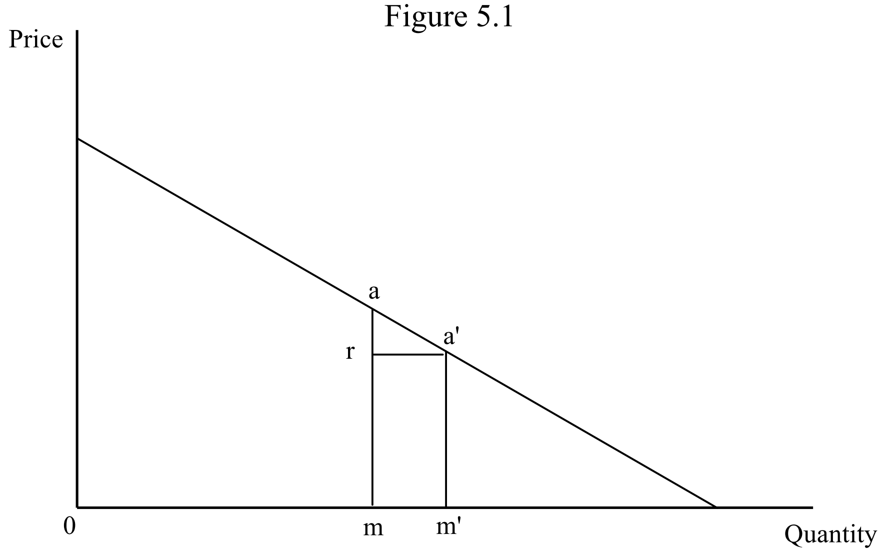
Marshall introduced the notion of price elasticity of demand in the following manner:

> We have seen that the only universal law as to a person's desire for a commodity is that it diminishes, other things being equal, with every increase in his supply of that commodity. But this diminution may be slow or rapid. If it is slow the price that he will give for the commodity

> will not fall much in consequence of a considerable increase in his supply of it; and a small fall in price will cause a comparatively large increase in his purchases. But if it is rapid, a small fall in price will cause only a very small increase in his purchases. In the former case his willingness to purchase the thing stretches itself out a great deal under the action of a small inducement: the elasticity of his wants, we may say, is great. In the latter case, the extra inducement given by the fall in price causes hardly any extension of his desire to purchase: the elasticity of his demands is small. If a fall in price from say 16d. to 15d. per lb. of tea would much increase his purchases, then a rise in price from 15d. to 16d. would much diminish them. That is, when the demand is elastic for a fall in price, it is elastic also for a rise. And as with the demand of one person so with that of a whole market. And we may say generally: The *elasticity* (or *responsiveness*) *of demand* in a market is great or small according as the amount demanded increases much or little for a given fall in price, and diminishes much or little for a given rise in price. [Marshall 1972: 86]

To formalize the above discussion, consider the following demand curve:

Figure 5.1



The elasticity of demand is (mm'/om)/(ar/am). If it is greater than, equal to, or less than |-1| then demand is elastic, unitary elastic, and inelastic respectively.

To make clearer the notion of the price elasticity of demand, Marshall stated that because there existed different income classes in society, the notion could only be understood in the context of a single income class. This is because the consumers' response to a price change—at this point a market demand curve is only being considered—is based solely on their diminishing marginal utility schedule. If, on the other hand, the demand of two or more income classes is under consideration, the response of quantity demanded to a change in the demand price will be based on an "income" effect as well as on diminishing marginal utility. In addition, Marshall acknowledged that there are graduations of income in a single income class which, in theory, would confound the notion of the price elasticity of demand; however, he assumed these minor subdivisions away. By restricting the notion of price elasticity of demand to a single homogeneous income class, Marshall explicitly acknowledged that his conception of the demand curve is not only independent of "income effects" but also not compatible with them.

Giffen Good Paradox

[rewrite; find history and put in footnote]

Stigler, in his 1947 article, "Notes on the History of the Giffen Paradox," argued that the paradox was a last-minute addition to the *Principles* since it stands in bold conflict with Marshall's law of demand. However, the Giffen Good Paradox does not conflict with the law of demand because it violates the assumption of given marginal utility of money:

> …as Sir. R. Giffen has pointed out, a rise in the price of bread makes so large a drain on the resources of the poorer labouring families and raises so much the marginal utility of money to them, that they are forced to curtail their consumption of meat and the more expensive farinaceous foods: and, bread being still the cheapest food which they can get and will take, they

> consume more, not less of it. But such cases are rare; when they are met with, each must be treated on its own merits. [Marshall 1972: 109–10]

Consumer Surplus

[not completed]

Critical Observations of Marshall's Analysis of Demand

Marshall's demand price is a indirect measure of a good's marginal utility to the consumer and hence an indirect measure of the consumer's subjective estimate of the good's use-value. Thus the given *wants* are the primal determinants of the demand price in that if a good does not satisfy any particular want of the consumer it is valueless. This raises the issue of the coincidence of given wants and given goods; that is to say, what forces exist to ensure that economic goods exist? The answer is that production adjusts to the given wants, assuming that given factors of production and given technology are appropriate—more on this below. As a result, the possibility exists where a consumer's wants are not satisfied by any of the existing goods. Marshall also stated that the law of diminishing marginal utility is part of human nature. While it may seem a plausible statement, its principle task is to ground the demand curve in nature. That is, since the law of diminishing marginal utility is the basis of the downward sloping demand curve, the latter is natural because the former is natural. As a result, the activities taken to satisfy wants cannot disturb the slope of the demand curve. Taking these two points together, we find that the demand price is a natural price, that is a price grounded in the individual's natural psyche.

By assuming the marginal utility of money as constant, Marshall ruled out the conventional income effect. The reason for doing so can, on the one hand, be traced to his desire to ensure that the demand curve sloped downward. On the other hand, the reason can be traced to the relationship between income and wants. Marshall argued that in the "higher study of consumption" activities taken to satisfy existing wants create new wants, or, more generally, progress as measured in terms of incomes bring civilization and hence new wants. Consequently, the concepts of efficiency, rationality, and equilibrium are undermined. Hence Marshall decided to concern himself only with natural wants; but to do so meant assuming away the income effect. That is, the income effect represents, in a very primitive way, progress and civilization and hence the creation of new wants and/or the alternation of old wants by the very activities undertaken to satisfy the old wants. Thus to prevent the undermining of his demand analysis, Marshall assumed the marginal utility of money as constant and did away with the potentially destructive income effect.

In constructing the demand curve for the individual (or the market), a comparative static analysis is used. Hence the points on the demand curve are a series of simultaneous alternative maximum demands for the quantities associated with each price. Thus, the demand curve is constituted for only a single act of exchange at a point in economic time. Once the exchange is consummated, the demand curve disappears and the consumer leaves the market. To reconstitute the demand curve, the consumer needs a new money income; but as long as the demand analysis takes money income as given and without explaining its origin, the demand curve can not in fact be reconstituted. Thus Marshall's demand analysis is explicitly restricted to a single point in economic time, a property that prevents it from being able to handle economic phenomena that exist through time. [more development here]

CHAPTER 6

MODERN UTILITY AND PREFERENCE THEORY

From Marshall to Hicks or Cardinal to Ordinal Utility

[needs to be done]

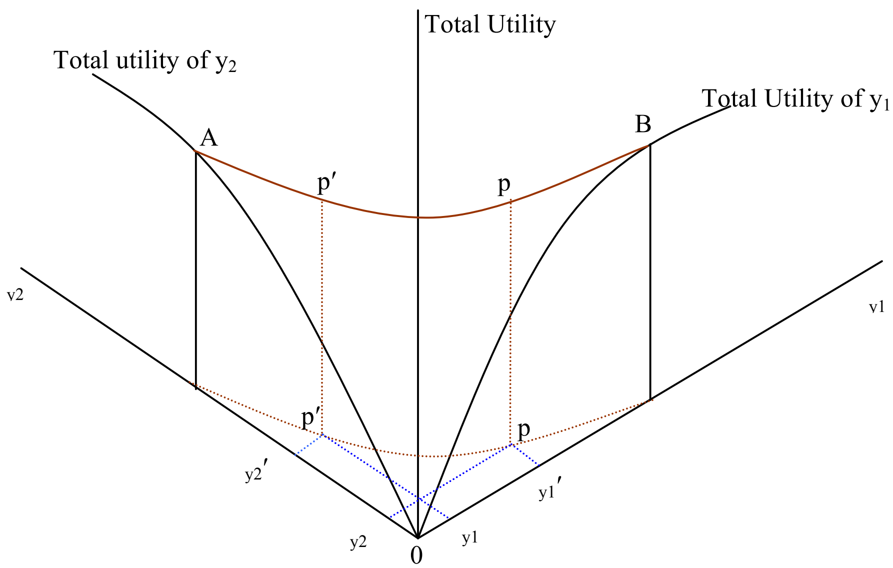
Hicks, Ordinal Utility, and Consumer Equilibrium

The advent of modern demand analysis came in the 1930s with the adoption of a generalized utility function in which the consumption of one good could affect the consumption of other goods and an ordinal approach to utility. That is, a utility function of the following form, $U = \mu(y_1, y_2, \ldots, y_n)$, recognizes the possibility that the utility derived from consuming Y_i may or may not be independent of the quantities of Y_j consumed, $i,j = 1, \ldots, n$ and $i \neq j$. By extending the utility function to this form, economists are not only able to deal with *substitutes* and *complements* goods, but also to explicitly and rigorously handle *substitution* and *income effects*. The generalized utility function also eliminated the need to have a cardinal measure utility and replaced it with an ordinal measure of utility. However, it should be noted that Marshall did not measure utility or even think it was measurable:

> It has already been agreed that desire cannot be measured directly, but only indirectly by the outward phenomena which they give rise: and that in those cases with which economics is chiefly concerned the measure is found in the price which a person is willing to pay for the fulfillment or satisfaction of his desire. [Marshall 1972: 78]

John Hicks, in his *Value and Capital*, starts his analysis of utility and preferences with a generalized utility function in which the individual knows how much utility he would derive from any given set of quantities of goods. Assuming that the individual prefers greater total utility to less, a scale of preferences is derived which for any two sets of goods, the consumer can say whether one is preferred to the other or whether he is indifferent between them. The locus of the set of goods that have the same total utility and hence are indifferent in the eyes of the consumer are called *indifferent curves*. For a better understanding of indifference curves, consider the following utility map:

Figure 6.1



Given quantities y_1 and y_2, the total utility received by the individual is pp; if the quantities change to y_1' and y_2' while the total utility remains unchanged, then total utility can be represented by p'p'. Hence it is easily seen that the curve AP'PB represents the locus of combinations of y_1 and y_2 that give the same total utility. Thus marginal utility theory can be captured by indifference curves; but in doing so, some of the original data is left behind. In particular, one thing left behind is the need to know the individual's utility map, and, hence, the need to measure utility in absolute, numerical terms. Instead the only thing needed to be known is the individual's indifference map, which only indicates the individual's preference for one particular set of goods over another set, and the sets of goods to which the individual is indifferent. In this manner, given wants are defined as a given *scale of preferences*.

In developing the indifference curve and scale of preferences, Hicks did not question the conceptual foundation of wants – that is being given to the analysis and independent of the activities used to satisfy them. Rather like Marshall, he took wants as given for economic purposes and saw economics as the study of the activities undertaken to satisfy these wants. Finally, Hicks used the concept of utility to connect the scale of preferences to wants, hence making the scale of preferences natural and given to the analysis. Thus, utility, by providing the motive for concrete choices has an irremovable function in modern demand theory.[4]

Given the indifference curve, its slope and shape can now be analyzed. The slope represents the amount of y_1 that is needed by the individual in order to compensate him for the loss of a small unit of y_2. Now the gain in utility got by gaining such an amount of y_1 equals the amount of y_1 gained times the marginal utility of y_1; the loss of utility got from losing the corresponding amount of y_2 equals the amount of y_2 lost times the marginal utility of y_2 (so long as the quantities are small). Therefore, since the gain equals the loss, the slope of the curve equals:

$$\frac{\text{amount of } y_2 \text{ lost}}{\text{amount of } y_1 \text{ gained}} = \frac{\text{marginal utility of } y_1}{\text{marginal utility of } y_2}$$

Hicks called the slope of the indifference curve the *marginal rate of substitution* (MRS_{21}) between the two goods. Although the MRS is negative, the question still remains as to the sign of its derivative. If it is positive, the indifference curve is convex to the origin; if it is negative, it is concave. Hicks **assumed** that indifference curves were *strictly convex* to the origin, thus making the derivative of the MRS positive. He called the derivative the *diminishing marginal rate of substitution* (DMRS) and provided the following intuitive understanding of it:[5]

> Suppose we start with a given quantity of goods, and then go on increasing the amount of Y_1 and diminishing that of Y_2 in such a way that the consumer is left neither better off nor worse off on balance; then the amount of Y_2 which has to be subtracted in order to set off a second unit of Y_1

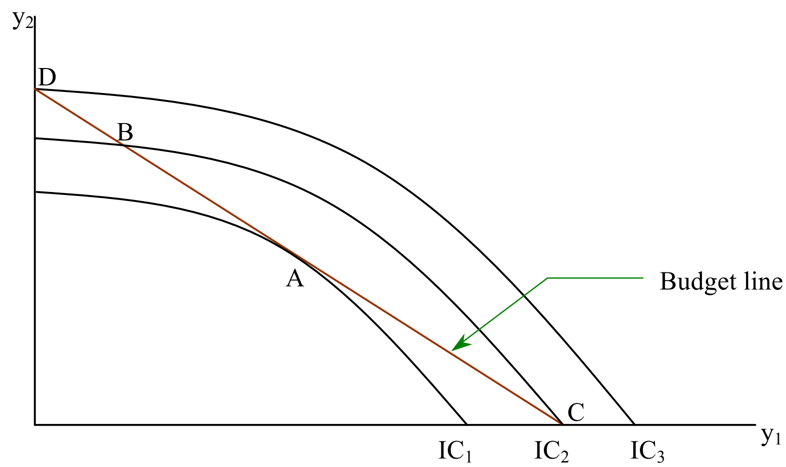
[4] See Wong (1978) for further discussion of Hicks and demand theory.
[5] Since the MRS is negative, a diminishing MRS must be positive.

will be less than that which has to be subtracted in order to set off the first unit. In other words, the more Y_1 is substituted for Y_2, the less will be the MRS of Y_1 for Y_2 (Hicks 1946: 20–21).

The necessity for the assumption will become evident once consumer equilibrium is introduced.

With the indifference curve in hand, Hicks proceeded to delineate *consumer equilibrium*. First he introduced the budget line that is defined as the locus of combinations of y_1 and y_2 that can be purchased if the entire money income (which is given) is spent. Its slope is the negative of the price ratio. Hicks then stated that through any point on the budget line, an indifference curve will pass through it. If the budget line and the indifference curve intersect the point will not be a utility maximizing equilibrium one; utility maximizing equilibrium occurs only when the indifference curve is tangent to the budget line. The reason is that at any intersection point, the individual can move along the budget line in one direction or another and reach an indifferent curve that has a higher total utility. It is only when the budget line is tangent to the indifference curve that utility is maximized. At such a point the $MRS_{21} = -MU_1/MU_2 = -p_1/p_2$ and the consumer will be in equilibrium. Now the reason for the strict convexity assumption can be broached. Consider the following strictly concave indifference curves:

Figure 6.2



At equilibrium point A, $MU_1/MU_2 = p_1/p_2$ and the consumer is minimizing his utility given money income. However, A is an unstable equilibrium since any movement away from it will increase the consumer's total utility; and the utility maximizing position will be point D, a corner solution of the furthest out indifference curve. Hence for consumer equilibrium to be stable and to be utility maximizing, the derivative of the MRS must be negative.

Formal Presentation of Consumer Equilibrium for a Two-Good Utility Function

Utility Function

The utility function is depicted as $U = \mu(y_1, y_2)$ where $y_1, y_2 > 0$. The utility function is assumed to have a first derivative with respect to y_1 and y_2:

$$\frac{\partial U}{\partial y_1} = \frac{\partial \mu(y_1, y_2)}{\partial y_1} = \mu_1 > 0 \text{ which is the marginal utility of } Y_1.$$

$$\frac{\partial U}{\partial y_2} = \frac{\partial \mu(y_1, y_2)}{\partial y_2} = \mu_2 > 0 \text{ which is the marginal utility of } Y_2.$$

A second derivative is also assumed to exist:

$$\frac{\partial^2 U}{\partial y_1^2} = \frac{\partial^2 \mu(y_1, y_2)}{\partial y_1^2} = \mu_{11} \underset{>}{\leq} 0 \quad \text{which is the rate of change in the marginal utility of } Y_1 \text{ due to a change in the quantity of } Y_1.$$

$$\frac{\partial^2 U}{\partial y_2^2} = \frac{\partial^2 \mu(y_1, y_2)}{\partial y_2^2} = \mu_{22} \underset{>}{\leq} 0 \quad \text{which is the rate of change in the marginal utility of } Y_2 \text{ due to a change in the quantity of } Y_2.$$

$$\frac{\partial^2 U}{\partial y_1 \partial y_2} = \frac{\partial^2 U}{\partial y_2 \partial y_1} = \frac{\partial^2 \mu(y_1, y_2)}{\partial y_1 \partial y_2} = \frac{\partial^2 \mu(y_1, y_2)}{\partial y_2 \partial y_1} = \mu_{12} = \mu_{21} \underset{>}{\leq} 0$$[6]

The signs of μ_{11} and μ_{22} depend on the nature of the utility function. If the utility function is strictly concave $\mu_{11} < 0$ and $\mu_{22} < 0$; however, if the utility function is strictly quasi-concave, μ_{11} and μ_{22} can be positive, negative, or zero.

Indifference Curve

The indifference curve is denoted as: $U^o = \mu(y_1, y_2)$, where U^o is a constant. To obtain the marginal rate of substitution, $U^o = \mu(y_1, y_2)$ is totally differentiated:

$$\frac{\partial \mu(y_1, y_2)}{\partial y_1} dy_1 + \frac{\partial \mu(y_1, y_2)}{\partial y_2} dy_2 = 0 \text{ or}$$

$$\frac{dy_2}{dy_1} = -\frac{\mu_1}{\mu_2} = -\frac{\text{marginal utility of } Y_1}{\text{marginal utility of } Y_2} = MRS_{21}.$$

Finally to obtain the diminishing marginal rate of substitution:

$$\frac{\partial MRS_{21}}{\partial y_1} = \frac{\partial(-\mu_1/\mu_2)}{\partial y_1} > 0 \text{ because of the assumption of strictly convex indifference curves or the law of diminishing marginal utility.}$$

Consumer Equilibrium

Given money income, M, the prices of the two goods, p_1 and p_2, and invoking the assumption that all money income is spent, the budget constraint is $M = p_1y_1 + p_2y_2$, the budget line is

[6] This result is due to *Young's Theorem*: let $y = f(x_1, x_2)$ have second-order partials that exist and are continuous, then $f_{12} = f_{21}$. In economics $\mu_{12} = \mu_{21}$ indicates that the effect of a change in the quantity of Y_1 consumed on the marginal utility of Y_2 is the same as the effect of a change in the quantity of Y_2 consumed on the marginal utility of Y_1

$y_2 = M/p_2 – (p_1/p_2)y_1$, and its slope is $–p_1/p_2$. Assuming utility maximization subject to a budget constraint, the Lagrangian function is:

$$L = \mu(y_1, y_2) + \lambda(M - p_1y_1 - p_2y_2).$$

The first order conditions (FOC) for utility maximization are:

$$L_1 = \frac{\partial\mu(y_1, y_2)}{\partial y_1} - \lambda p_1 = \mu_1 - \lambda p_1 = 0$$

$$L_2 = \frac{\partial\mu(y_1, y_2)}{\partial y_2} - \lambda p_2 = \mu_2 - \lambda p_2 = 0$$

$$L_\lambda = M - p_1y_1 - p_2y_2 = 0.$$

Rearranging the FOC, we find that $-\mu_1/\mu_2 = –p_1/p_2 = MRS_{21}$ and $M = p_1y_1 + p_2y_2$ or the equilibrium conditions for consumer equilibrium that maximizes utility—see Figure 6.3. To see if a utility maximization position has in fact been reached, the second order conditions are needed:

$L_{11} = \mu_{11}$; $L_{12} = \mu_{12}$; $L_{1\lambda} = -p_1$
$L_{21} = \mu_{21}$; $L_{22} = \mu_{22}$; $L_{2\lambda} = -p_2$
$L_{\lambda 1} = -p_1$; $L_{\lambda 2} = -p_2$; $L_{\lambda\lambda} = 0$.

Putting this into a bordered Hessian matrix and taking its determinant, we have:

$$\begin{vmatrix} \mu_{11} & \mu_{12} & -p_1 \\ \mu_{21} & \mu_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} > 0$$ since the indifference curves are assumed to be strictly convex.

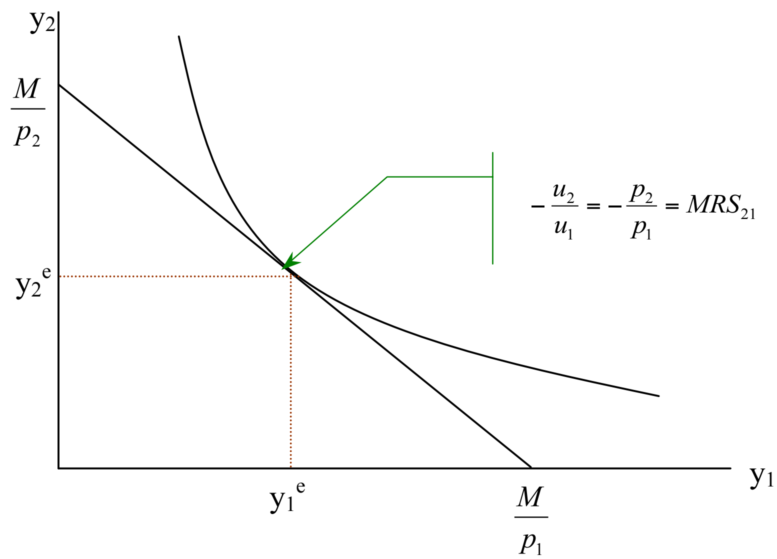
Solving the FOC, we get the equilibrium demand functions for y_1 and y_2 and the equilibrium marginal utility of money λ:

$$y_1^e = f_1(p_1, p_2, M)$$

$$y_2^e = f_2(p_1, p_2, M)$$

$$\lambda^e = f_\lambda(p_1, p_2, M).$$

Figure 6.3



Preferences and Utility Function

Economists begin their analysis by assuming a utility function exists for each consumer and has the general form: $U = \mu(y_1, y_2, \ldots, y_n)$. Although each of the n goods is associated with a dimension in number space, not all of number space is used, that is, y_i cannot be just any value. Rather three properties are imposed on the vector of goods—$\mathbf{y} = (y_1, \ldots, y_n)$—so as to make it amendable for analysis:

(1) a good may not be characterized by a strictly negative number, hence $\mathbf{y} > 0$ that is it can have no negative components;

(2) divisibility—let $\mathbf{y^i} = (y_1^i, \ldots, y_n^i)$ be a bundle of goods available to the consumer, then any bundle of the form $\alpha\mathbf{y^i} = (\alpha y_1^i, \ldots, \alpha y_n^i)$ where $0 \le \alpha \le 1$ can be extracted; and

(3) the good vector is bounded from below—$\mathbf{y} = (0,\ldots,0)$, but not from above.

Preference Structure

Following Hicks, economists assumed that a consumer decides whether he/she prefers one bundle of goods to another or is indifferent between them. Such an assumption implies that a consumer has complete knowledge of all possible bundles of goods, that is, a consumer's knowledge is not limited by his/her social class, income class, or personal experience—in fact these characteristics are irrelevant. The criterion for preferring or being indifferent is whether one bundle of goods gives more total utility or gives the same total utility as another bundle of goods. That is, let $U_i = \mu_i(\mathbf{y^i})$, then we have the following relationships:

(1) if $U_i > U_j$, then $\mathbf{y^i}$ is preferred to $\mathbf{y^j}$;

(2) if $U_i < U_j$, then $\mathbf{y^j}$ is preferred to $\mathbf{y^i}$; or

(3) if $U_i = U_j$, then $\mathbf{y^i}$ and $\mathbf{y^j}$ are equally preferred.

With this notation, we are able to state the three basic axioms needed to establish the existence of a continuous utility function:

Axiom of comparability: for any $\mathbf{y^i}$ and $\mathbf{y^j}$, the consumer is able to say $\mathbf{y^i}$ is preferred to or is equally preferred to $\mathbf{y^j}$, or $\mathbf{y^i} \geq \mathbf{y^j}$. [need to deal with framing effect]

Axiom of transitivity: using $\geq$ as preferred to or is indifferent to, we have $\mathbf{y^i} \geq \mathbf{y^j}$ and $\mathbf{y^j} \geq \mathbf{y^z}$, then $\mathbf{y^i} \geq \mathbf{y^z}$.

Axiom of continuity: the set of good bundles not preferred to $\mathbf{y^i}$ and the set of good bundles preferred to $\mathbf{y^i}$ are both closed in the good space for any $\mathbf{y^i}$. That is, take any $\mathbf{y^i} > \mathbf{y^j}$, this says that it is possible to let $\mathbf{y^j}$ come close enough to $\mathbf{y^i}$ for $\mathbf{y^i}$ to be indifferent to $\mathbf{y^j}$.[7]

[7] This axiom excludes the possibility of lexicographic preference ordering in which goods are ranked like words in a dictionary. Under such a preference ordering the utility function that has indifference curves does not exist; or more directly, such a preference ordering is inconsistent with indifference curves. [more discussion]

The first two axioms ensure that the consumer's preferences are consistent. All three axioms provide sufficient conditions for the existence of a real-value utility function which is a continuous function of the quantities consumed such that $U_i \geq U_j$ when $\mathbf{y^i} \geq \mathbf{y^j}$.

Utility Function

However, the axioms do not ensure the existence of a utility function whose properties would permit utility maximization subject to a budget constraint. This is remedied by the following axioms:

Axiom of dominance (or monotonicity or non-satiation): there is some good y_i such that $\mathbf{y^i} > \mathbf{y^j}$ if $y_i^i > y_i^j$ and $y_i^i = y_i^j$ for all $i \neq j$. That is, there is some good, say the ith good that the consumer would always prefer to consume more of, that is with which s/he is never sated. Hence, $U = \mu(y_1, \ldots, y_n)$ is a strictly increasing function of the quantities consumed (hence the term monotonicity). This assumption guarantees that all bundles in the preferred subset exhausts the consumer's income. That is, preferred consumption bundles are represented by points on the budget line. This follows from the fact that any bundle that does not exhaust the consumer's income is not maximizing utility. The assumption also helps ensure that there is sufficient demand to create relatively scarce factor inputs.[8]

Axiom of strictly quasi-concave utility function: a strictly concave utility function has two properties, the first being that marginal utility of the ith good declines and the second being that the Hessian matrix of second partials is negative ensuring utility maximization. However, utility maximization can also be achieved with the weaker assumption of a strictly quasi-concave utility function. In this case, the direction of movement of marginal utility of the ith good can be increasing, decreasing or constant while the Hessian matrix of second partials is negative ensuring utility maximization. Because the Hessian matrix is negative definite in both cases,

[8] This point will be discussed in more detail later in Part IV.

either functions will produce strictly convex indifference curves. Thus utility maximization is obtained by assuming a particular form of the utility function that also produces strictly convex indifference curves.[9]

Axiom of differentiability: it is assumed that the utility function is twice differentiable so that calculus can be used.

Indifference Curve (do I need this?)

The indifference curve can be denoted as: $U^o = \mu(y_1, \ldots, y_n)$, where U^o is a constant. The marginal rate of substitution represents the maximal rate at which the consumer's consumption of a good j can be reduced, without reducing his/her utility, when the consumption of the ith good is increased. Because the MRS is a two-good concept, its delineation becomes somewhat complicated when the utility function contains more than two goods. So in this case let

$U^o = \mu(y_1,\ldots, y_{j-1}, \phi_u(\mathbf{y^j}), y_{j+1}, \ldots, y_n)$ where $y_j = \phi_u(y_1,\ldots, y_{j-1}, y_{j+1}, \ldots, y_n) = \phi_u(\mathbf{y^j})$.

Now we can differentiated U^o with respect to y_i ($i \neq j$):

$$\frac{\partial\mu(\mathbf{y})}{\partial y_i} + \frac{\partial\mu(\mathbf{y})}{\partial\phi_u(\mathbf{y^j})} \times \frac{\partial\phi_u(\mathbf{y^j})}{\partial y_i} = 0.$$

Rearranging, we have $MRS_{ji} = -\dfrac{\partial\phi_u(\mathbf{y^j})}{\partial y_i} = \dfrac{\partial\mu(\mathbf{y})/\partial y_i}{\partial\mu(\mathbf{y})/\partial\phi_u(\mathbf{y^j})}$ or

$-\dfrac{\partial y_j}{\partial y_i} = \dfrac{\partial\mu(\mathbf{y})/\partial y_i}{\partial\mu(\mathbf{y})/\partial y_j} < 0$ because of the assumption of a strictly quasi-concave utility function.[10]

[9] As noted above, for the consumer to have a unique utility maximization position, its indifference curve map must consist solely of strictly convex indifference curves. That is, for example, if the indifference curve is quasi-concave, it would have a "flat" portion that coincided with the budget line at more than one point. Hence there would exist many different bundles of goods that would maximize the consumer's utility, given the consumer's income and prices. The axiom of strictly quasi-concave utility function is invoked to eliminate this possibility.

[10] Hick (1946) assumed the law of diminishing marginal rate of substitution so that the indifference curves would be strictly convex. However, when more than two goods inhabit the utility function, it is not sufficient for strict convexity of the indifference curves (or surfaces). The strict convexity requirement is fulfilled by assuming a strictly quasi-concave utility function. Consequently the law of diminishing marginal rate of substitution is not relevant or needed in modern utility and demand theory.

Budget Constraint, Consumer Choice, and Consumer Equilibrium

To arrive at the utility maximization equilibrium, four assumptions are made. The first is that the consumer is endowed with a given positive amount of money income, M;[11] the second is that there exists a fixed positive price for each of the n goods the consumer can possibly buy; the third is that the consumer must spend his/her entire income on the consumption of goods; and the last is that for any given set of positive prices and money income, the consumer will chose a consumption bundle—called the optimal consumption bundle—on the budget line. Assuming utility maximization subject to a budget constraint, the Lagrangian function is:

$L = \mu(y_1, \ldots, y_n) + \lambda(M - p_1y_1 - \ldots - p_ny_n).$

The first order conditions (FOC) for utility maximization are:

$$L_1 = \frac{\partial\mu(\mathbf{y})}{\partial y_1} - \lambda p_1 = \mu_1 - \lambda p_1 = 0$$

$$L_2 = \frac{\partial\mu(\mathbf{y})}{\partial y_2} - \lambda p_2 = \mu_2 - \lambda p_2 = 0$$

…………………………….

$$L_n = \frac{\partial\mu(\mathbf{y})}{\partial y_n} - \lambda p_n = \mu_n - \lambda p_n = 0$$

$L_\lambda = M - p_1y_1 - \ldots - p_ny_n = 0.$

Rearranging the FOC, we find that $\lambda = \mu_i/p_i$, $-\mu_i/\mu_j = -p_i/p_j = MRS_{ji}$, and $M = p_1y_1 + \ldots + p_ny_n$ or the equilibrium conditions for consumer equilibrium that maximizes utility. To see if a utility maximization position has in fact been reached, the second order conditions are needed:

$L_{11} = \mu_{11}; \ldots; L_{1n} = \mu_{1n}; L_{1\lambda} = -p_1$

……………………………….

[11] The origin of the money income is not considered (it is derived from given endowments—but issues here). It should also be noted that if the activities undertaken to satisfy a consumer's want generated and determined his/her income, then the independence of wants, activities, and money income is seriously questioned. [more-references]

$L_{n1} = \mu_{n1}; \ldots; L_{nn} = \mu_{nn}; L_{n\lambda} = -p_n$
$L_{\lambda 1} = -p_1; \ldots; L_{\lambda n} = -p_n; L_{\lambda\lambda} = 0.$

Putting this into a bordered Hessian matrix and taking its determinant, we have:

$$\begin{vmatrix} \mu_{11} & \ldots & \mu_{1n} & -p_1 \\ & \ldots\ldots\ldots\ldots & & \\ \mu_{n1} & \ldots & \mu_{nn} & -p_n \\ -p_1 & \ldots & -p_n & 0 \end{vmatrix} > 0.$$

This result emerges because of the axiom of strictly quasi-concave utility function ensures that the determinant of the bordered Hessian is negative definite. Consequently, the consumer equilibrium position is a local maximum as well as a global maximum. In addition, $\mu_{ij} = \mu_{ji}$ because of Young's theorem and $\mu_{ii} < 0$ if the utility function is strictly concave or $\mu_{ii} >\leq 0$ if the utility function is strictly quasi-concave. Finally, solving the FOC, we get the equilibrium demand functions for $y_1, \ldots, y_n$ and the equilibrium marginal utility of money λ:

$$y_1^e = f_1(p_1, \ldots, p_n, M)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_n^e = f_2(p_1, \ldots, p_n, M)$$
$$\lambda^e = f_\lambda(p_1, \ldots, p_n, M).$$

CHAPTER 7

CONSUMER DEMAND THEORY

The last chapter ended with solving the FOC to get the equilibrium demand functions for $y_1, \ldots, y_n$ and the equilibrium marginal utility of money λ:

$$y_1^e = f_1(p_1, \ldots, p_n, M)$$
$$\cdots\cdots\cdots\cdots$$
$$y_n^e = f_n(p_1, \ldots, p_n, M)$$
$$\lambda^e = f_\lambda(p_1, \ldots, p_n, M).$$

The objective of this chapter is to examine the properties of the demand functions with respect to the substitution effect, income effect, and the shape of the demand curve; the law of demand; and the market demand curve, price elasticity of demand, and cross effects. To carry this out, the methodological procedure of *comparative statics* (which is based on the *ceteris paribus* method is used). Applying comparative statics to explaining the shape of the demand curve or cross effects, the unknowns are the equilibrium quantities demanded, y_i^e; the functional relationships are the demand functions derived from the first order conditions and are based on the underlying preference structure embodied in the utility function; and the parameters are prices and money income. The objective is to determine the direction of change of y_i^e with respect to the change in a parameter. [do a little more here]

Shape of the Demand Curve and the Slutsky Equation

To us comparative statics to explain the shape of the demand curve, the equilibrium demand functions obtained from solving FOC are substituted back into the first order conditions and then differentiated with respect to p_i. Thus, we have the following:

$$L_1 = \frac{\partial \mu(\mathbf{y}^e)}{\partial y_1} - \lambda^e p_1 \equiv 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$L_n = \frac{\partial \mu(\mathbf{y}^e)}{\partial y_n} - \lambda^e p_n \equiv 0$$

$$L_\lambda = M - p_1 y_1^e - \ldots - p_n y_n^e \equiv 0$$

where $\mathbf{y^e} = (y_1^e, \ldots, y_n^e)$.

Since the consumer's demand curve for the ith good is under investigation, each first order condition is differentiated with respect to p_i:

$$L_{1i} = \frac{\partial^2\mu(\mathbf{y^e})}{\partial y_1^2}\frac{\partial y_1^e}{\partial p_i} + \ldots + \frac{\partial^2\mu(\mathbf{y^e})}{\partial y_1 \partial y_n}\frac{\partial y_n^e}{\partial p_i} - \frac{\partial \lambda^e}{\partial p_i} p_1 \equiv 0$$

……………………………………………………..

$$L_{ii} = \frac{\partial^2\mu(\mathbf{y^e})}{\partial y_i \partial y_1}\frac{\partial y_1^e}{\partial p_i} + \ldots + \frac{\partial^2\mu(\mathbf{y^e})}{\partial y_i^2}\frac{\partial y_i^e}{\partial p_i} + \ldots + \frac{\partial^2\mu(\mathbf{y^e})}{\partial y_i \partial y_n}\frac{\partial y_n^e}{\partial p_i} - \frac{\partial \lambda^e}{\partial p_i} p_i - \lambda^e \equiv 0$$

……………………………………………………

$$L_{ni} = \frac{\partial^2\mu(\mathbf{y^e})}{\partial y_n \partial y_1}\frac{\partial y_1^e}{\partial p_i} + \ldots + \frac{\partial^2\mu(\mathbf{y^e})}{\partial y_n^2}\frac{\partial y_n^e}{\partial p_i} - \frac{\partial \lambda^e}{\partial p_i} p_n \equiv 0$$

$$L_{\lambda i} = - p_1 \frac{\partial y_1^e}{\partial p_i} - \ldots - p_i \frac{\partial y_i^e}{\partial p_i} - y_i^e - \ldots - p_n \frac{\partial y_n^e}{\partial p_i} \equiv 0.$$

Putting the above into matrix form and letting $\mu_{ij} = \partial^2\mu(\mathbf{y^e})/\partial y_i \partial y_j$:

$$\begin{bmatrix} \mu_{11} & \ldots & \mu_{1n} & -p_1 \\ \ldots & \ldots & \ldots & \ldots \\ \mu_{i1} & \ldots & \mu_{in} & -p_i \\ \ldots & \ldots & \ldots & \ldots \\ \mu_{n1} & \ldots & \mu_{nn} & -p_n \\ -p_1 & \ldots & -p_n & 0 \end{bmatrix} \begin{bmatrix} \partial y_1^e/\partial p_i \\ \ldots \\ \partial y_i^e/\partial p_i \\ \ldots \\ \partial y_n^e/\partial p_i \\ \partial \lambda^e/\partial p_i \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \ldots \\ \lambda^e \\ \ldots \\ 0 \\ y_i^e \end{bmatrix}$$

Now using *Cramer's Rule*, $\partial y_i^e/\partial p_i$ can be solved for:

$$\frac{\partial y_i^e}{\partial p_i} = \frac{\lambda^e D_{ii} + y_i^e D_{n+1i}}{D}.$$

This is called the *Slutsky equation*. To determine its sign, the signs of D_{ii} and D_{n+1i} have to be ascertained since D is negative definite. In doing so, the concepts of *substitution effect* and *income effect* are utilized. That is, the own-price derivative of the demand function can be decomposed into a substitution effect and into an income effect. The *substitution effect* is that part of the variation in quantity demanded that is due to the change in relative prices with the consumer consuming more of the ith good whose relative price has decreased and less of the goods whose relative prices have increased while maintaining the same level of total utility. The *income effect* is that part of the variation in quantity demanded that is due to the change in real income as a result of the change in p_i with money income and all other prices remaining constant. The sum of both effects is equal to the variation in quantity demanded of the ith good due to a change in p_i.

Substitution Effect

To delineate the substitution effect, a Lagrangian function is formed in which expenditures are *minimized* subject to achieving a given level of total utility:

$L = p_1y_1 + \ldots + p_ny_n + \varphi[U^0 - \mu(y_1, \ldots, y_n)]$.

FOC:

$$L_1 = p_1 - \varphi\frac{\partial\mu(\mathbf{y})}{\partial y_1} = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$L_n = p_n - \varphi\frac{\partial\mu(\mathbf{y})}{\partial y_n} = 0$$
$$L_\varphi = U^0 - \mu(\mathbf{y}) = 0$$

Since the utility function is strictly quasi-concave, the equilibrium position derived from the first order conditions is a minimum equilibrium position. Solving the first order conditions, we get a *Hicksian* or *compensated* demand functions:

$$y_1^u = f_1^u(p_1, \ldots, p_n, U^0)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_n^u = f_n^u(p_1, \ldots, p_n, U^0)$$
$$\varphi^u = f_\varphi^u(p_1, \ldots, p_n, U^0).$$

In these demand functions, the quantity demanded is dependent only upon relative prices and a given level of utility; hence $\partial y_I^u/\partial p_i$ represents the change in quantity demanded due only to a change in relative prices originating from a change in p_i. To show this, $y_1^{\varphi}, \ldots, y_n^{\varphi}, \varphi^u$ are substituted back into the FOC and then differentiated with respect to p_i and we get:

$$0 - \varphi^u\left[\frac{\partial^2\mu(\mathbf{y^u})}{\partial y_1^2}\frac{\partial y_1^u}{\partial p_i} + \ldots + \frac{\partial^2\mu(\mathbf{y^u})}{\partial y_1 \partial y_n}\frac{\partial y_n^u}{\partial p_i}\right] - \frac{\partial\varphi^u}{\partial p_i}\frac{\partial\mu(\mathbf{y^u})}{\partial y_1} \equiv 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$1 - \varphi^u\left[\frac{\partial^2\mu(\mathbf{y^u})}{\partial y_i \partial y_1}\frac{\partial y_1^u}{\partial p_i} + \ldots + \frac{\partial^2\mu(\mathbf{y^u})}{\partial y_i^2}\frac{\partial y_1^u}{\partial p_i} + \ldots + \frac{\partial^2\mu(\mathbf{y^u})}{\partial y_i \partial y_n}\frac{\partial y_n^u}{\partial p_i}\right] - \frac{\partial\varphi^u}{\partial p_i}\frac{\partial\mu(\mathbf{y^u})}{\partial y_i} \equiv 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$0 - \frac{\partial\mu(\mathbf{y^u})}{\partial y_1}\frac{\partial y_1^u}{\partial p_i} - \ldots - \frac{\partial\mu(\mathbf{y^u})}{\partial y_n}\frac{\partial y_n^u}{\partial p_i} \equiv 0.$$

Putting into matrix form and letting $\mu_{ij}^u = \partial\mu(\mathbf{y^u})/\partial y_i \partial y_j$:

$$\begin{bmatrix} -\mu_{11}^u\varphi & \ldots & -\mu_{1n}^u\varphi & -\mu_1 \\ \cdots & \cdots & \cdots & \cdots \\ -\mu_{i1}^u\varphi & \ldots & -\mu_{in}^u\varphi & -\mu_i \\ \cdots & \cdots & \cdots & \cdots \\ -\mu_1 & \ldots & -\mu_n & 0 \end{bmatrix} \begin{bmatrix} \partial y_1^u/\partial p_i \\ \cdots \\ \partial y_i^u/\partial p_i \\ \cdots \\ \partial\varphi^u/\partial p_i \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \cdots \\ -1 \\ \cdots \\ 0 \end{bmatrix}$$

Using Cramer's Rule, $\partial y_i^u/\partial p_i$ can be solved for:

$$\frac{\partial y_i^u}{\partial p_i} = \frac{(-1)(D_{ii}^u)}{D^u} < 0$$

since D^u and D_{ii}^u are negative by virtue of the assumed strictly quasi-concave utility function. Taking a closer look at D_{ii}^u and D^u we find the following:

$D_{ii}^u = (-\varphi)^{-1}(1/\varphi)^{-n+1}(\varphi)^{-2}(-1)^{-n+1}D_{ii}$ where D_{ii} is the D_{ii} found in the Slutsky equation; and

$D^u = (-\varphi)^{-1}(1/\varphi)^{-n}(\varphi)^{-2}(-1)^{-n}D$ where D is the D found in the Slustky equation.

Substituting, we have

$$\frac{\partial y_i^u}{\partial p_i} = \frac{(-1)[(-\varphi)^{-1}(1/\varphi)^{-n+1}(\varphi)^{-2}(-1)^{-n+1}D_{ii}]}{(-\varphi)^{-1}(1/\varphi)^{-n}(\varphi)^{-2}(-1)^{-n}D} = \frac{D_{ii}}{\varphi D} = \frac{\lambda D_{ii}}{D} < 0$$

since the FOC can be written as $\varphi = p_i/\partial\mu(\mathbf{y})/\partial y_i = p_i/\mu_i$ which is the inverse of λ. Thus $\partial y_i^u/\partial p_i$ represents the substitution effect and it is always negative. So by substituting into the Slutsky equation we have:

$$\frac{\partial y_i^e}{\partial p_i} = \frac{\partial y_i^u}{\partial p_i} + \frac{y_i^e D_{n+1i}}{D}.$$

Income Effect

To delineate the income effect, the FOC can be differentiated with respect to money income M:

$$L_{1M} = \frac{\partial^2\mu(\mathbf{y}^e)}{\partial y_1^2}\frac{\partial y_1^e}{\partial M} + \ldots + \frac{\partial^2\mu(\mathbf{y}^e)}{\partial y_1 \partial y_n}\frac{\partial y_n^e}{\partial M} - \frac{\partial\lambda^e}{\partial M} p_1 \equiv 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$L_{\lambda M} = 1 - p_1\frac{\partial y_1^e}{\partial M} - \ldots - p_n\frac{\partial y_n^e}{\partial M} \equiv 0.$$

Putting the above into matrix form and letting $\mu_{ij} = \partial^2\mu(\mathbf{y}^e)/\partial y_i \partial y_j$:

$$\begin{bmatrix} \mu_{11} & \ldots & \mu_{1n} & -p_1 \\ \ldots & \ldots & \ldots & \ldots \\ \mu_{n1} & \ldots & \mu_{nn} & -p_n \\ -p_1 & \ldots & -p_n & 0 \end{bmatrix} \begin{bmatrix} \partial y_1^e/\partial M \\ \ldots \\ \partial y_n^e/\partial M \\ \partial\lambda^e/\partial M \end{bmatrix} = \begin{bmatrix} 0 \\ \ldots \\ 0 \\ -1 \end{bmatrix}$$

Now using *Cramer's Rule*, $\partial y_i^e/\partial M$ can be solved for:

$$\frac{\partial y_i^e}{\partial M} = \frac{(-1)D_{n+1i}}{D} \gtrless 0$$ since D_{n+1i} is not a border-preserving principle minor.

Taking this result and substituting it into the Slutsky equation we have

$$\frac{\partial y_i^e}{\partial p_i} = \frac{\partial y_i^u}{\partial p_i} - y_i^e\frac{\partial y_i^e}{\partial M} \gtrless 0$$

where $\frac{\partial y_i^u}{\partial p_i}$ is the substitution effect whose sign is known, but the overall sign is not known because the sign of the income effect, $y_i^e\frac{\partial y_i^e}{\partial M}$ can be positive or negative.

Before examining these results, let us take a look at $\partial y_i^e/\partial M$—which represents the change in the quantity demanded of good i when money income changes. Since prices are given, the demand function for good i can be written as $y_i^e = f_i^*(M)$. Such a demand function is called an *Engel curve* and its first derivative has either a positive or negative sign. Using y_i^e and M as the initial equilibrium position, it is possible to examine the first derivative result more closely using the *income elasticity of demand* that is defined as $\partial y_i^e M/\partial M y_i^e$:

(1) if $\partial y_i^e/\partial M = 0$ and $\frac{\partial y_i^e}{\partial M}\frac{M}{y_i^e} = 0$ then the quantity of good i does not change with money income--see curve I in Figure 7.1.

(2) if $\partial y_i^e/\partial M > 0$, $\partial^2 y_i^e/\partial M^2 < 0$, and $0 < \frac{\partial y_i^e}{\partial M}\frac{M}{y_i^e} < 1$ the good i is a *normal good*—see curve II in Figure 7.1.

(3) if $\partial y_i^e/\partial M > 0$, $\partial^2 y_i^e/\partial M^2 = 0$, and $0 < \frac{\partial y_i^e}{\partial M}\frac{M}{y_i^e}$ then the Engel curve is linear. If it intercepts the y_i^e axis at a point greater than zero, its income elasticities are smaller than one but tend towards one as money income increases. If it has a negative intercept with the y_i^e axis, then the income elasticity of demand is greater than one and tends to one as money income increases—see curve III in Figure 7.1.

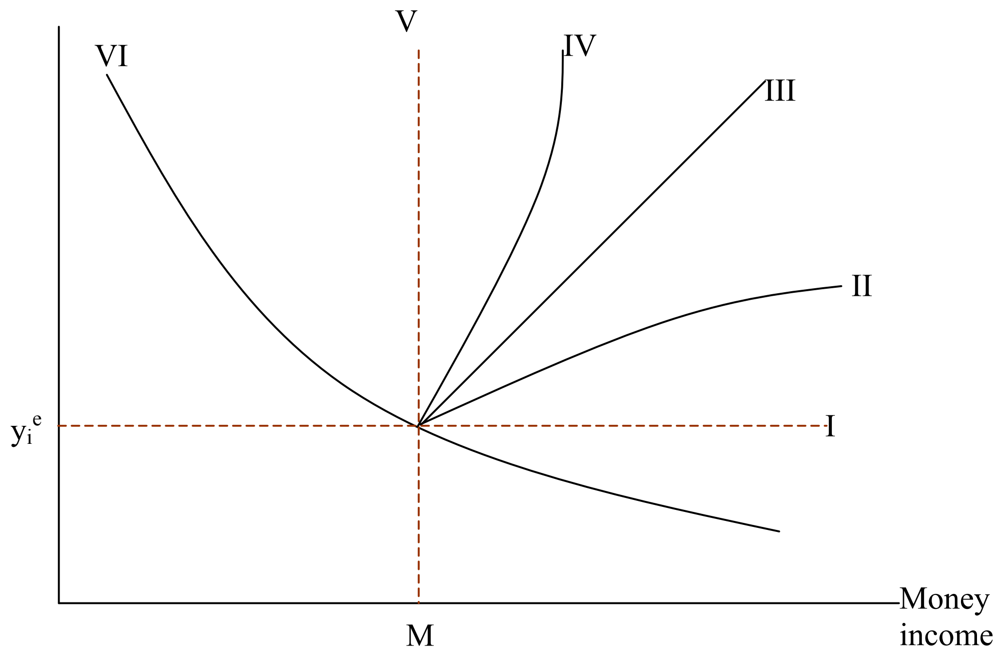
(4) if $\partial y_i^e/\partial M > 0$, $\partial^2 y_i^e/\partial M^2 > 0$, and $1 < \frac{\partial y_i^e}{\partial M}\frac{M}{y_i^e}$ the good i is a *superior* good—see curve IV in Figure 7.1.

(5) if $\partial y_i^e/\partial M = \infty$ the good represents the ultimate superior good whose income elasticity of demand is infinity—see curve V in Figure 7.1.

(6) if $\partial y_i^e/\partial M < 0$, $\partial^2 y_i^e/\partial M^2 < 0$, and $0 > \frac{\partial y_i^e}{\partial M}\frac{M}{y_i^e}$ the good i is a *inferior* good since its income elasticity of demand is negative—see curve VI in Figure 7.1.

A curve whose income elasticity of demand is one throughout has special significance in that the consumer's pattern of consumption does not vary with variations in money income. That is, if a consumer's Engel curve for the ith good has an income elasticity of one, then, say, a ten percent increase in money income will result in a ten percent increase in the consumption of the ith good. If all the consumer's Engel curves had unitary income elasticity, then his relative consumption pattern would remain invariant as money income changes. Such a situation implies that wants and activities are independent of each other.[12]

Figure 7.1



Shape of the Consumer Demand Curve and the Law of Demand

We are now in a position to provide a more comprehensive understanding of the shape of the consumer demand curve. The consumer demand function for the ith good is denoted as $y_i^e = f_i(p_1,\ldots,p_n, M)$; and the change in the equilibrium quantity demanded due to a change in p_i is denoted by the Slutsky equation:

$\frac{\partial y_i^e}{\partial p_i} = \frac{\partial y_i^u}{\partial p_i} - y_i^e\frac{\partial y_i^e}{\partial M} \gtrless 0$ because of the unknown sign of the income effect.

[12] See the discussion of homothetic utility functions in chapters 8 and 9.

Because the sign of the Slutsky equation can be positive, negative, or zero, the consumer's demand curve can slope downward, upward, or vertical. More specifically, if the y_i is a normal or superior good, the income effect is negative thus reinforcing the substitution resulting in a downward sloping demand curve. However, if y_i is an inferior good, then the income effect is positive which diminishes the negative impact of the substitution effect; and if the income effect is significantly positive, then $\partial y_i^e / \partial p_i > 0$. In this case, y_i is called a Giffen good.[13] Thus, because of the indeterminate sign of the income effect, it cannot be assumed *apriori* that the consumer's demand curve for any good obeys Marshall's general law of demand or *any law of demand*. [more]

Market Demand Curve and its Properties

The usual method of deriving the market demand curve is to add the individual consumer demand horizontally; and the resulting 'aggregate' market demand curve is assumed to behave as if it represents the choices of a single utility maximizing consumer. That is, the market demand curve is conceived as qualitatively identical to the individual consumer demand curve, meaning that it has a determinant relationship between price and quantity that is based on the income and substitution effects. However, in general it is not possible to add up consumer demand curves to get a market demand with such properties—but this will be dealt with in chapter 9. For the moment, we shall delineate the properties of demand curves as if they apply to both consumer and market demand curves.

Properties of Demand Functions

Homogenity – demand functions are homogeneous of degree zero in income and prices; so if all prices and incomes are multiplied by a positive constant k, the quantity demanded remains unchanged. Using the Lagrangian function:

$L = \mu(y_1, \ldots, y_n) + \lambda k(M - p_1 y_1 - \ldots - p_n y_n)$.

[13] History of the Giffen good is needed.

The first order conditions (FOC) for utility maximization are:

$$L_1 = \frac{\partial\mu(\mathbf{y})}{\partial y_1} - \lambda k p_1 = \mu_1 - \lambda k p_1 = 0$$

$$L_2 = \frac{\partial\mu(\mathbf{y})}{\partial y_2} - \lambda k p_2 = \mu_2 - \lambda k p_2 = 0$$

……………………………….

$$L_n = \frac{\partial\mu(\mathbf{y})}{\partial y_n} - \lambda k p_n = \mu_n - \lambda k p_n = 0$$

$$L_\lambda = k(M - p_1 y_1 - \ldots - p_n y_n) = 0.$$

Solving the FOC for the demand functions, we find that k drops out and we get the following demand functions:

$$y_I^e = f_I(p_1, \ldots, p_n, M).$$

That is, the quantity demanded is unaffected by a proportionate increase in money income and prices.

Adding-up – the budget constraint has to be satisfied over the range of variations of prices and income. The demand functions have therefore to be such that the sum of expenditures on the different goods equals total money income. The property can be expressed in the following way:

$$\underline{\partial\,[M \equiv p_1 y_1^e + \ldots + p_n y_n^e]}/\partial M \equiv 1 \equiv p_1 \partial y_1^e/\partial M + \ldots + p_n \partial y_n^e/\partial M.$$

Defining $p_i \partial y_i^e/\partial M$ as the *marginal propensity to consume* of the ith good, the adding up property says that the marginal propensities to consume must sum up to one.

Price Elasticity of Demand

The price elasticity of demand is defined as the percentage change in quantity demand divided by the percentage change in the price and can be represented by:

$$\left|\frac{p_i \partial y_i^e}{\partial p_i y_i^e}\right| = |\in_{ii}^e|$$

If $|\in_{ii}^e| > 1$ then demand is elastic; if $|\in_{ii}^e| < 1$ then demand is inelastic; and if $|\in_{ii}^e| = 1$ then demand is unitary elastic. To understand the basis for the different elasticities, multiply the Slutsky equation by Mp_i/My_i^e to get the "price elasticity" Slutsky equation:

$$\frac{p_i \partial y_i^e}{\partial p_i y_i^e} = \frac{p_i \partial y_i^u}{\partial p_i y_i^e} - \frac{p_i y_i^e}{M} \times \frac{M \partial y_i^e}{\partial M y_i^e} \text{ or}$$

$$\in_{ii}^e = \in_{ii}^u - k_i \in_{im}$$

where $\in_{ii}^e$ is the price elasticity of demand of the response of y_i^e to a change in p_i, holding M and all other prices constant;

$\in_{ii}^u$ is the price elasticity of demand of the response of y_i^e to a change in p_i, holding utility and all other prices constant;

$k_i = p_i y_i^e/M$ is the share of the consumer's budget spent on good i; and

$\in_{im}$ is the income elasticity of demand for good i.

If k_i is very small, then $\in_{ii}^e$ is predominately determined by $\in_{ii}^u$ and whether good i is a superior, normal, or inferior good does not really matter. However, if k_i is large enough to matter, then as long as $\in_{im} > 1$or good i is a superior good, then $|\in_{im}| > 1$; and if $0 < \in_{im} < 1$ and $\in_{im} > [1 - |\in_{ii}^u|]/k_i$ then $|\in_{ii}^e| > 1$. However, if $\in_{im} < [1- |\in_{ii}^u|]/k_i$ then $|\in_{ii}^e| < 1$, so it is possible for a normal good to be associated with inelastic demand. Finally, if good i is an inferior good with $\in_{im} < 0$, then $|\in_{ii}^e| < 1$ or it is inelastic; and if good I is a Giffen good, then it does not have a price elasticity of demand in any meaningful sense. The price elasticity Slutsky equation also provides insight on the shape of the demand curve. If k_i is very small, then irrespective of whether good i is a superior, normal, or inferior good, the substitution effect will dominate and the demand curve will slope downward. Only if k_i is relatively large will $\in_{im}$ have any impact. So if $\in_{im} \geq 0$ and $\in_{ii}^u < 0$, or as long as good i is a normal or superior good, the consumer's demand curve will slop downward; or if $\in_{im} < 0$, but if $|\in_{im}| < |\in_{ii}^u|/k_i$ then $\in_{ii}^e < 0$ or the consumer's demand curve for good i will slope downward as long as the good is not too inferior.

However if $\in_{im} < 0$ and $|\in_{im}| > |\in_{ii}^u|/k_i$, then $\in_{ii}^e > 0$, or the consumer's demand curve for good i will slope upward if it is sufficiently inferior. As noted above, such a good is called a Giffen good. Thus the income elasticity of demand plays the principle role in theoretically (scientifically) upsetting the "general law of demand" (this section needs to be re-examined).

Cross Effects, Substitutes, and Complements

So far only the relationship between the quantity demanded of the ith good and its own price has been examined. However using the matrix equation on pages 31, the relationship between the quantity demanded of good j and the price of the ith good is:

$$\frac{\partial y_j^e}{\partial p_i} = \frac{\lambda^e D_{ij} + y_I^e D_{n+1j}}{D} \gtrless 0$$

since both minors are not principle minors and the last term is not a border preserving minor and $\partial y_j^e/\partial p_i$ is called a *cross effect* as it involves the effect of the change in price of one good on the quantity demanded of another good. Cross effect can be classified in the following manner:

(1) if $\partial y_j^e/\partial p_i > 0$ the goods are *gross substitutes* – that is the rise in price of the ith good increases the quantity demanded of the jth good;

(2) if $\partial y_j^e/\partial p_i < 0$ the goods are *gross complements* – that is the rise in price of the ith good decreases the quantity demanded of the jth good; or

(3) if $\partial y_j^e/\partial p_i = 0$ the goods are *gross independents* – that is the rise in price of the ith good does not affect the quantity demanded of the jth good.

This classification of substitutes and complements is not satisfactory for neoclassical microeconomic theory because of the unpredictable income term. That is, because the income term is not related to the consumer's preference structure, it could "in theory' turn goods that are "obvious" substitutes into complements and vice versa. Therefore a second classificatory scheme is developed based on compensated (or Hicksian) demand functions. Using the matrix equation on page 33, the relationship between the quantity demanded of good j and the price of the ith good is:

$$\frac{\partial y_j^u}{\partial p_i} = \frac{(-1)(D_{ij}^u)}{D^u} = \frac{\lambda^e D_{ij}}{D} \gtrless 0$$

since D_{ij} is not a principle minor.

Net cross effects can be classified in the following manner:

(1) $\partial y_j^u/\partial p_i > 0$ the goods are net substitutes;

(2) $\partial y_j^u/\partial p_i < 0$ the goods are net complements; or

(3) $\partial y_j^u/\partial p_i = 0$ the goods are net independents.

It should be noted that in a two good utility function, the goods are always net substitutes, but they can be gross complements only if a Giffen good situation exists.

CHAPTER 8

SPECIAL TOPICS IN CONSUMER DEMAND THEORY

[Need to do something about the importance of special topics; that is what they consist of and why neoclassical economists think that they are important. Special topics largely work inside the model of demand—see Varian for working with models and whether this is a particular way of creating knowledge.]

Indirect Utility Function, Cost Minimizing Functions, and Duality

Indirect Utility Function

The utility function used to derive the demand curve is denoted as $U = \mu(y_1, \ldots, y_n)$. By solving the first order conditions, equilibrium demand functions – $y_1^e, \ldots, y_n^e$ – are obtained. Now, by substituting them back into the utility function, we get $U = \mu(y_1^e, \ldots, y_n^e) = g\,(p_1,\ldots, p_n, M) = g(\mathbf{p}, M)$ where $\mathbf{p} = (p_1,\ldots, p_n)$. This new utility function, $U = g(\mathbf{p}, M)$, is called a *indirect utility function* since the level of utility depends indirectly on prices and money income. The properties of the indirect utility function are (Varian 1992: 102-3]:

(a) since the utility function is continuous so is $g(\mathbf{p}, M)$.

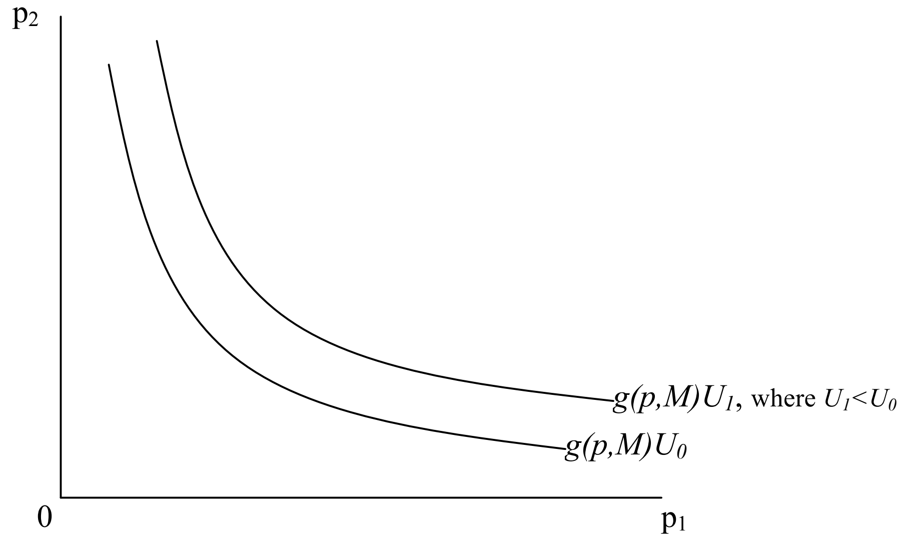
(b) $g(\mathbf{p}, M)$ is non-increasing prices; if $\mathbf{p'} > \mathbf{p}$, then $g(\mathbf{p'}, M) \leq g(\mathbf{p}, M)$. The non-increasing property results from the fact that different price vectors can give the same total utility. However $g(\mathbf{p}, M)$ is strictly decreasing for an increase in a single price if all other remain constant.

(c) $g(\mathbf{p}, M)$ is non-decreasing in money income; so if money income increases so will total utility because of the non-satiation principle.

(d) $g(\mathbf{p}, M)$ is strictly quasi-convex in $\mathbf{p}$.

(e) the level curves of $g(\mathbf{p}, M)$ are strictly convex; that is the *indirect indifference curves* are strictly convex in terms of prices:

Figure 8.1



An indirect indifference curve shows those combinations of p_1 and p_2 that have the same total utility; hence the convexity property indicates that at a relatively high p_2, relatively small changes in p_1 are needed to compensate for large changes in p_2. In addition, the closer the indirect indifference curves are to the origin, the higher their utility.

(f) $g(\mathbf{p}, M)$ is homogeneous of degree 0 in $(\mathbf{p}, M)$.

To maximize utility with an indirect utility function, the Lagrangian function is $L = g(\mathbf{p}, M) + \psi(M - p_1 y_1^e - \dots - p_n y_n^e)$. The first order conditions are:

$$L_1 = \partial g(\mathbf{p}, M)/\partial p_1 - \psi y_1^e = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$L_n = \partial g(\mathbf{p}, M)/\partial p_n - \psi y_n^e = 0$$

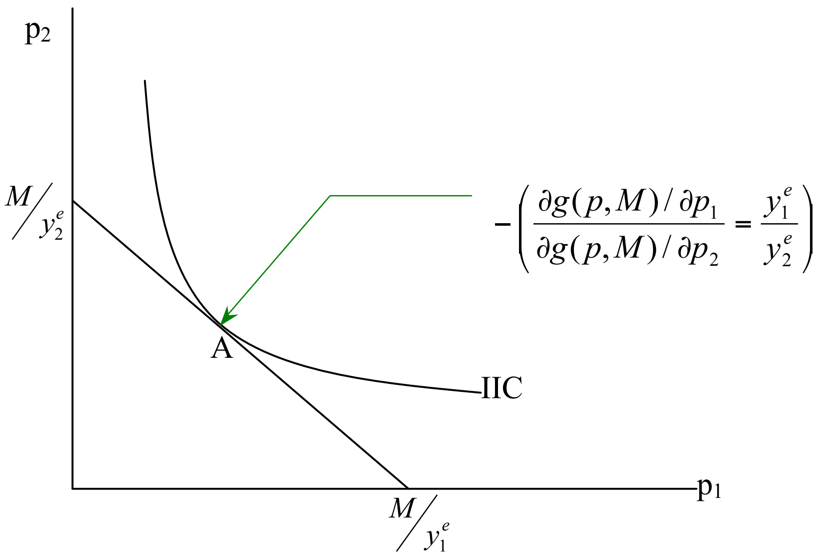
$$L_\psi = M - p_1 y_1^e - \dots - p_n y_n^e = 0.$$

Conditions for equilibrium is

$$\frac{\partial g(\mathbf{p}, M)/\partial p_i}{\partial g(\mathbf{p}, M)/\partial p_j} = \frac{y_i^e}{y_j^e}$$

which means, in a two good-two price system, that the slope of the indirect indifference curve, $-(\partial g(\mathbf{p}, M)/\partial p_1)/(\partial g(\mathbf{p}, M)/\partial p_2)$ equals the slope of the good budget line, $-y_1^e/y_2^e$:

Figure 8.2



Roy's Identity

Assuming that $g(\mathbf{p}, M)$ is a well-behaved indirect utility function, then the equilibrium demand function for the ith good can be obtained in the following manner [Varian 1992: 106]:

$$y_i^e = f_i(\mathbf{p}, M) = -\frac{\partial g(\mathbf{p}, M)/\partial p_i}{\partial g(\mathbf{p}, M)/\partial M}.$$ [14]

---

[14]Proof: working with the above FOC, we have

$$\frac{\partial g(\mathbf{p}, M)/\partial p_i}{\psi} = y_i^e = 0$$

Now differentiating the Lagrangian function with respect to money income, we get

$L_m = \partial g(\mathbf{p}, M)/\partial M + \psi = 0$ or $-\partial g(\mathbf{p}, M)/\partial M = \psi$.

Substituting we get $-\dfrac{\partial g(\mathbf{p}, M)/\partial p_i}{\partial g(\mathbf{p}, M)/\partial M} = y_i^e$.

Cost Minimizing Functions

So noted in the previous chapter, to derive the substitution effect a Lagrangian function is set up in which expenditure is minimized subject to achieving a given level of total utility:

$L = p_1y_1 + \ldots + p_ny_n + \varphi[U^0 - \mu(y_1, \ldots, y_n)]$.

From this, compensated demand functions are derived and which are of the form

$y_i^u = f_i^u(p_1, \ldots, p_n, U^0)$.

Now, by substituting the compensated demand functions into $M = p_1y_1 + \ldots + p_ny_n$ we get

$M = p_1y_1^u + \ldots + p_ny_n^u = c(\mathbf{p}, M)$ which is called a *cost minimizing function* (or *expenditure function*) or simply *cost function*. The cost function relates a give level of total utility to the minimal amount of income needed to achieve it; that is money income is a function of prices and a give level of total utility. Properties of the cost function are the following (Varian 1992: 104-5]:

(a) it is continuous in prices and the first and second derivatives with respect to P exist everywhere.

(b) it is increasing in U^o, non decreasing in **p**, and increasing in at least one price. These properties follow immediately from the non-satiation assumption. At given prices the consumer has to spend more to be better off, while increases in prices require at least as much expenditure to remain as well off.

(c) it is concave in prices.

(d) it is homogenous of degree one in prices; that is, if prices double, twice as much money income in needed to stay on the same indifference curve.

Hotelling's Theorem

Since the equilibrium demand function and the compensated demand function are 'duals' of each other, so are to the indirect utility function and the cost function. If the cost function is differentiable, then compensated demand functions can be obtained from it by differentiating with respect to prices: $\partial c(\mathbf{p}, U^o)/\partial p_i = y_i^u = f_i^u(\mathbf{p}, U^o)$.[15] [Varian ??]

Duality

Duality is defined as the existence of two logical systems characterized by certain interrelationships. The essence of a dual system is its correspondence between concepts in one logical system and concepts in the other which allows us to derive a correspondence between results in one system and results in the other. So in a dual system, there is generally a correspondence between variables in one system and variables in the other, between functions in one system and functions in the other, and between operations in one system and operations in the other. Duality theorems, then, say that if a certain proposition can be proved and if it can be shown that a proposition in the alternative system is a dual to that one, then the dual proposition holds as well. This is important because it often makes it easier to prove the proposition that is dual to the first one.

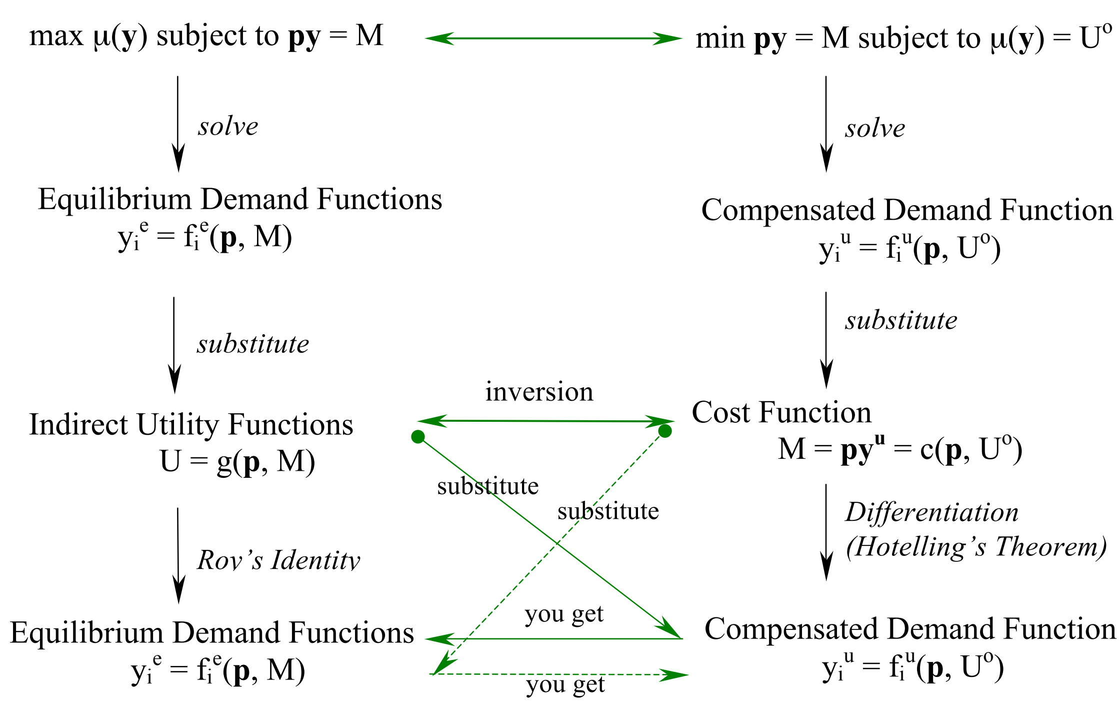
The essential feature of the duality approach in demand theory is a change of variables. Preference and utility are originally defined over quantities as the objects of choice and this "primal" formulation, of U in terms of **y,** is certainly the most obvious. However, if the consumer faces a linear budget constraint, its position, as defined by **p** and money income, determines maximum attainable utility so that U can just as well be regarded as a function of money income and prices (the indirect

[15] Proof: consider an arbitrary vector of prices $\mathbf{p^o}$, a level of total utility U^o, and the corresponding vector of compensated demand functions, $\mathbf{y^u}$. Then for any other price vector $\mathbf{p}$, defined the function $Z(\mathbf{p})$ by $Z(\mathbf{p}) = \sum_i^n p_i y_i^u - c(\mathbf{p}, U^o)$. Since $\mathbf{y^u}$ is not necessarily optimal for $\mathbf{p}$, the cost of $\mathbf{y^s}$ at $\mathbf{p}$ must always be at least as great as the cost of the optimal vector at $\mathbf{p}$, that is, $c(\mathbf{p}, U^o)$. Hence Z is always greater than or equal to zero. But we know that Z is equal to zero, or attains its minimum, when $\mathbf{p}$ is equal to $\mathbf{p^o}$. Hence, if the derivative exists at $\mathbf{p^o}$, $\underline{\partial Z(\mathbf{p^o})}/\partial p_i = y_i^u - \underline{\partial c(\mathbf{p^o}, U^o)}/\partial p_i = 0$ or $y_i^u = \underline{\partial c(\mathbf{p^o}, U^o)}/\partial p_i$ which since $\mathbf{p^o}$ is quite arbitrary, proves Hotelling's Theorem.

utility function) or, inversely, money income can be regarded as a function of U and prices (the cost function). To summarize the above analysis of the indirect utility function and the cost function and to indicate their duality, let us look at Figure 8.3:

Figure 8.3

Duality



[Deaton and Muellbauer 1999: 38, 41]

Special Utility Functions

Separable Utility Function

Instead of having the form $U = \mu(y_1, \ldots, y_n)$ a separable utility function has the following form $U = \mu[f_a(y_1, \ldots, y_l), f_b(y_{i1}, \ldots, y_j), \ldots, f_z(y_m, \ldots, y_n)]$. There are two kinds of separable utility functions: weakly separable and strongly separable utility functions.

Weakly Separable Utility Function

A weakly separable utility function has the form of $U = \mu[f_a(y_1,\ldots,y_i),\ldots,f_z(y_{j+m},\ldots,y_n)]$. The condition for such a function is that the marginal rate of substitution between any two goods in the same group are independent of the value of any good in any other group:

$$\frac{\partial f/\partial y_{ai}}{\partial f/\partial y_{aj}} \neq 0, \quad \text{but} \quad \frac{\partial}{\partial y_{zn}} \left| \frac{\partial f/\partial y_{ai}}{\partial f/\partial y_{aj}} \right| = 0$$

where $\partial f/\partial y_{ai} = [\partial\mu/\partial f_a][\partial f_a/\partial y_{ai}]$ and $\partial f/\partial y_{aj} = [\partial\mu/\partial f_a][\partial f_a/\partial y_{aj}]$

The usefulness of this utility function can be found in constructing *utility trees* in which goods are grouped according to their general utility and then each group is broken down into specific goods with specific utilities.[16]

Strongly Separable (or Additive) Utility Function

A strongly separable utility function is one in which each good is separated from each other as in the case of Marshall's utility function.: $U = \mu_1(y_1)+\ldots+\mu_n(y_n)$ which can either be strictly convex or strictly quasi-concave. Setting up the Lagrangian function, we have $U = \mu_1(y_1) +\ldots+ \mu_n(y_n) + \lambda(M - \mathbf{py})$. Taking the FOC we have

$$L_1 = \frac{\partial\mu_1(y_1)}{\partial y_1} - \lambda p_1 = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$L_\lambda = M - \mathbf{py} = 0$$

Now solving for the equilibrium demand functions we get

$$y_i^e = f_i(\mathbf{p},M),\ i = 1,\ldots,n$$

$$\lambda^e = f_\lambda(\mathbf{p},M).$$

Substituting y_i^e and λ^e back into the FOC and then differentiating with respect to M, we get

[16] Statement about how separable utility function has ideas that are relevant to heterodox approaches to consumer demand and choice—use-value replaces utility..

$$\frac{\partial^2 \mu_1(y_1^e)}{\partial y_1^2}\frac{\partial y_1^e}{\partial M} - \frac{\partial \lambda^e}{\partial M} p_1 \equiv 0$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{\partial^2 \mu_n(y_n^e)}{\partial y_n^2}\frac{\partial y_n^e}{\partial M} - \frac{\partial \lambda^e}{\partial M} p_n \equiv 0$$

$$1 - \frac{\partial y_1^e}{\partial M}(p_1) - \dots - \frac{\partial y_n^e}{\partial M}(p_n) \equiv 0$$

Rearranging we get

$$\begin{bmatrix} \partial^2\mu_1(y_1^e)/\partial y_1^2 & 0 & \dots\dots & 0 & -p_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & & \partial^2\mu_n(y_n^e)/\partial y_n^2 & -p_n \\ -p_1 & -p_2 & \dots\dots & -p_n & 0 \end{bmatrix} \begin{bmatrix} \partial y_1^e/\partial M \\ \dots \\ \partial y_n^e/\partial M \\ \partial \lambda^e/\partial M \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \dots \\ 0 \\ -1 \end{bmatrix}$$

Using Cramer's Rule to solve for $\partial\lambda^e/\partial M$, we get

$$\frac{\partial \lambda^e}{\partial M} \equiv \frac{(-1)D_{n+1,n+1}}{D} \gtrless 0,$$ because $D_{n+1,n+1}$ is not a border preserving minor

Let us now return to

$$\frac{\partial^2 \mu_1(y_1^e)}{\partial y_1^2}\frac{\partial y_1^e}{\partial M} - \frac{\partial \lambda^e}{\partial M} p_1 \equiv 0$$

$$\dots\dots\dots\dots\dots\dots$$

$$\frac{\partial^2 \mu_n(y_n^e)}{\partial y_n^2}\frac{\partial y_n^e}{\partial M} - \frac{\partial \lambda^e}{\partial M} p_n \equiv 0$$

$$1 - \frac{\partial y_1^e}{\partial M}(p_1) - \dots - \frac{\partial y_n^e}{\partial M}(p_n) \equiv 0$$

and rearrange the first n equations in the following manner:

$$\frac{\partial y_1^e}{\partial M} \equiv \frac{p_1}{\mu_{11}}\frac{\partial \lambda^e}{\partial M}$$

$$\dots\dots\dots$$

$$\frac{\partial y_n^e}{\partial M} \equiv \frac{p_n}{\mu_{nn}}\frac{\partial \lambda^e}{\partial M}$$

where $\mu_{ii} = \partial^2\mu_i(y_i)/\partial y_i^2$

Working with the above and n + 1 equation $(\partial y_1^e/\partial M)p_1 + ... + (\partial y_n^e/\partial M)p_n \equiv 1$, we can arrive at the following results:

(1) If we assume a strictly concave strongly separable utility function then $\mu_{ii} < 0$ for all i = 1,..., n, then $\partial\lambda^e/\partial M < 0$ (that is the marginal utility of money is declining), since $D_{n+1,n+1}$ will always be the opposite sign of D. Consequently all $\partial y_i^e/\partial M > 0$ for i = 1,..., n. Therefore, all goods will be normal goods which, in turn, ensure that all demand curves slope downward.

(2) If we assume a strictly quasi-concave strongly separable utility function with increasing marginal utility of money ($\partial\lambda^e/\partial M > 0$), then we find that n-1 goods have diminishing marginal utility and are inferior goods – that is, if $\partial\lambda^e/\partial M > 0$ and $\mu_{ii} < 0$ for i = 1,..., n-1, then $\partial y_i^e/\partial M < 0$ for i = 1,..., n-1. However, at least the nth good must have increasing marginal utility and be a normal good.

Example: $U = \mu_1(y_1) + \mu_2(y_2) + \lambda(M - p_1y_1 - p_2y_2)$ where y_1 is a normal good with increasing marginal utility, the marginal utility of money is (λ) is increasing, and the utility function is strictly quasi-concave. So

FOC

$$\partial\mu_1(y_1)/\partial y_1 - \lambda p_1 = 0$$

$$\partial\mu_2(y_2)/\partial y_2 - \lambda p_2 = 0$$

$$M - p_1y_1 - p_2y_2 = 0$$

Second order conditions (in determinant form) are

$$\begin{vmatrix} \mu_{11} & 0 & -p_1 \\ 0 & \mu_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} = (\mu_{11})(-p_2^2) + (\mu_{22})(-p_1^2) > 0.$$

Given the above and $(\partial y_1^e/\partial M)p_1 + ... + (\partial y_1^e/\partial M)p_n \equiv 1$, we have:

$$\frac{\partial y_1^e}{\partial M} \equiv \frac{p_1}{\mu_{11}} \frac{\partial \lambda^e}{\partial M}$$

$$\frac{\partial y_2^e}{\partial M} \equiv \frac{p_2}{\mu_{22}} \frac{\partial \lambda^e}{\partial M}$$

Since $\mu_{11} > 0$, $\mu_{22} < 0$, and $\partial\lambda^e/\partial M > 0$ then $\partial y_1^e/\partial M > 0$ meaning that y_1 is a normal good and $\partial y_2^e/\partial M < 0$ meaning that y_2 is an inferior good.

Let us now consider the strongly separable utility function in terms of net complements and substitutes. Working with the Lagrangian function $\mathbf{py} + \varphi(U^0 - \sum_{i=1}^{n} \mu_i(y_i)$, we can derive the following compensated demand function:

$$y_i^u = f_i^u(\mathbf{p}\ U^0) \quad \text{and}$$
$$\varphi^u = f_\varphi^u(\mathbf{p}\ U^0)$$

Substituting back into the FOC, differentiating with respect to p_i, and putting in matrix form we get:

$$\begin{bmatrix} -\mu_{11}\varphi & 0 & 0 & -\mu_1 \\ \ldots & & & \\ 0 & -\mu_{ii}\varphi & 0 & -\mu_i \\ \ldots & & & \\ 0 & 0 & -\mu_{ii}\varphi & -\mu_n \\ -\mu_1 \ldots & -\mu_i \ldots & -\mu_n & 0 \end{bmatrix} \begin{bmatrix} \partial y_1^u/\partial p_i \\ \ldots \\ \partial y_i^u/\partial p_i \\ \ldots \\ \partial y_n^u/\partial p_i \\ \partial \mu^u/\partial p_i \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \ldots \\ -1 \\ \ldots \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial y_1^u}{\partial p_i} = \frac{(-1)D_{i,1}}{D} \gtrless 0$$

Let us inspect the first row further:

$$\mu_{11}\varphi(\partial y_1^u/\partial p_i) + \mu_1(\partial \varphi^u/\partial p_i) \equiv 0 \quad \text{or}$$

$$\frac{\partial y_1^u}{\partial p_i} = \frac{-\mu_1}{\mu_{11}\varphi} \frac{\partial \varphi^u}{\partial p_i} = \frac{-\mu_1}{\mu_{11}\varphi} \frac{\partial y_1^u}{\partial U_0}$$

because of the well-known reciprocity relationship, $\partial\varphi/\partial p_i = \partial y_i^u/\partial U_0$. Now we can conclude the following: if $\mu_{ii} < 0$ for all $i = 1,..., n$, then all goods are normal (that is $\partial y_i^u/\partial U_0 > 0$ for all $i = 1,..., n$), and hence all goods are net substitutes ($\partial y_j^u/\partial p_i > 0$). We can also conclude the following: if $\mu_{ii} > 0$, then $\partial y_i^u/\partial U_0 < 0$ hence $\partial y_j^u/\partial p_i > 0$. That is if the i^{th} good has increasing marginal utility, then the j^{th} good is inferior while they are net substitutes. More specifically each good $j = 1,..., n$ and $j \neq i$ is inferior and net substitutes with respect to good i. Finally, we find that $\mu_{jj} < 0$, $j = 1,..., n$, $j \neq i$ and $\partial y_j^u/\partial U_0 < 0$, $j = 1,..., n$, $j \neq i$, thus leading to $\partial y_j^u/\partial p_k < 0$ where $k \neq i,j$ – that is the inferior goods are not complements to each other.

Homogeneous Utility Function

A function $f(x_1,..., x_n)$ is said to be homogeneous of degree r if and only if $f(tx_1,..., tx_n) \equiv t^r f(x_1,..., x_n)$. Thus a homogeneous utility function of degree r is $U = \mu(ty_1,..., ty_n) \equiv t^r \mu(y_1,..., y_n)$ where r is a constant and r is a positive real number. The marginal rate of substitution of good i for good j:

$$MRS_{ij} = \frac{\partial\mu(t\mathbf{y})/\partial y_i}{\partial\mu(t\mathbf{y})/\partial y_j} \equiv \frac{t^r\partial\mu(\mathbf{y})/\partial y_i}{t^r\partial\mu(\mathbf{y})/\partial y_j} = \frac{\partial\mu(\mathbf{y})/\partial y_i}{\partial\mu(\mathbf{y})/\partial y_j}$$

Thus, the MRS_{ij} is invariant with respect to proportionate changes in consumption levels. It also follows that if a consumer is indifferent between two consumption bundles, he will also be indifferent between any other bundles that use the same multiple of the first pair. In other words, the slopes of the indifference curves are constant along a ray from the origin. [Need to put in an example in a footnote].

Homothetic Utility Function

A function is homothetic if it can be written in the form $U = \Phi[f(x_1,..., x_n)]$ where Φ is a positive monotonically increasing function and f is a homogeneous function. A utility function is said to be homothetic if for some function $\mu(y_1,..., y_n)$ which is homogeneous of degree one, we can write $U = \Psi[\mu(y_1,..., y_n)]$ which is a monotonically increasing function – that is $\partial\Psi/\partial y > 0$.[17] For such a utility function, the doubling of y will double total utility. In this case, each indifference curve is simply a

[17] A homogeneous function of degree one is also a homothetic function, but not the reverse. That is, all homogeneous functions of any degree can be transformed into a function of degree one, and all homogeneous functions of degree one are homothetic functions. However, not all homothetic functions are homogeneous functions.

magnified or reduced version of every other one. Thus, any ray from the origin will cut all the indifference curves at points where the slope is the same. This has two implications. First, because of the constant slope property, the *income-consumption path* and the associate Engel curve given by increasing money income with prices constant will be straight lines form the origin. The implication is that the composition of the budget is independent of total utility. Hence all the income elasticities of demand are unity.[18] The second implication is that the structure of the cost function becomes $M = c(\mathbf{p},U^0) = U^0 b(\mathbf{p})$. Consequently doubling money income will double total utility. This in turn implies that the indirect utility function can be written as $U = g(\mathbf{p}, M) = g(\mathbf{p})M$. Roy's identity then implies that the equilibrium functions take the form of $y_i^e = f_i^e(\mathbf{p}, M) = f_i^e(\mathbf{p})M$—that is, they are linear functions of income. The fact that the "income effects" take this special form is essential for arriving at a consistent aggregation of consumer demand curves to produce a "well-formed" market demand curve (see chapter 9). [Need more discussion here]

Revealed Preference

Revealed preference represents an attempt by economists to show that their preference and demand theory actually does explain consumer behavior in the real world (and without using the concept of utility).[19] To show this correspondence, economists initially posit consumption bundles consumed by individuals and then work back to the underlying utility function. Let us consider the following consumption bundles $\mathbf{y^1} = (y_1^1, \ldots, y_n^1)$ and $\mathbf{y^2} = (y_1^2, \ldots, y_n^2)$ and price vectors $\mathbf{p^1} = (p_1^1, \ldots, p_n^1)$ and $\mathbf{p^2} = (p_1^2, \ldots, p_n^2)$. Now if $\mathbf{p^1y^1} \geq \mathbf{p^1y^2}$ and $\mathbf{p^1y^1} = M$, then $\mathbf{y^1}$ is said to be revealed preferred to $\mathbf{y^2}$. This also implies that $\mathbf{p^2y^1} > \mathbf{p^2y^2}$. Thus we can now state the *weak axiom of revealed preference*:

[18] The income-consumption path is a locus of utility-maximizing bundles in which prices are fixed (hence have the same MRS = price ratio) while money income is varied.
[19] NEED MORE OF A HISTORICAL DISCUSSION HERE—Wong (1978).

Assume that $\mathbf{y^1}$ is revealed preferred to $\mathbf{y^2}$, that is, at some price vector $\mathbf{p^1}$, $\mathbf{y^1}$ is chosen, and $\mathbf{p^1y^1} \geq \mathbf{p^1y^2}$ and therefore $\mathbf{p^2y^1} > \mathbf{p^2y^2}$, so that $\mathbf{y^2}$ could have been chosen but was not. Then $\mathbf{y^2}$ will never be revealed preferred to $\mathbf{y^1}$.

The weak axiom ensures that all demand function $y_i^e = f_i(\mathbf{p}, M)$ are homogeneous of degree zero in all prices and money income; that all demand functions are single-valued, that is for any price-income vector $(\mathbf{p}, M)$ the consumer chooses a single point of consumption; and that the matrix $\partial y_i^e/\partial p_j$ ij = 1, …, n is negative semi-definite, that is $\partial y_i^e/\partial p_j \leq 0$. In turn this result ensures that $\partial y_i^e/\partial p_i < 0$.[20]

The weak axiom does not, however, ensure that $\partial y_i^e/\partial p_j = \partial y_j^e/\partial p_i$. Consequently, the weak axiom allows intransitivity of preferences to occur. If revealed preference is to be associated with the usual notions of consumers' preferences, we cannot allow the situation where $\mathbf{y^1}$ is preferred to $\mathbf{y^2}$ and $\mathbf{y^2}$ is preferred to $\mathbf{y^3}$ and then have $\mathbf{y^3}$ preferred to $\mathbf{y^1}$ cannot be allowed. Such intransitivity could not occur under the usual assumptions of utility analysis; yet this situation is precisely what can occur under the weak axiom. Thus a stronger axiom is needed: the *strong axiom of revealed preference*:

Let the bundle of goods purchased at price vector $\mathbf{p^i}$ be denoted $\mathbf{y^i}$. For any finite set of bundles $(\mathbf{y^1, \ldots, y^k})$, if $\mathbf{y^1}$ is revealed preferred to $\mathbf{y^2}$, $\mathbf{y^2}$ revealed preferred to $\mathbf{y^3}$, etc., …, $\mathbf{y^{k-1}}$ revealed preferred to $\mathbf{y^k}$, or, algebraically, if $\mathbf{p^1y^1} \geq \mathbf{p^1y^2}$, $\mathbf{p^2y^2} \geq \mathbf{p^2y^3}$, …, $\mathbf{p^{k-1}y^{k-1}} \geq \mathbf{p^{k-1}y^k}$, then $\mathbf{p^ky^k} < \mathbf{p^ky^{o;}}$ that is $\mathbf{y^k}$ is not revealed to $\mathbf{y^1}$.

Consequently, a set of individual demand function $y_i^e = f_I(\mathbf{p}, M)$, i = 1, …, n which are consistent with the strong axiom of revealed preference are derivable from the utility function. Thus with the strong axiom, we are assuming that the consumer's behavior as 'observed' in the market is consistent with preference and demand theory. In other words, the strong axiom is equivalent to the utility-

[20] For proof of these results, see Silberberg and Suen (2001).

maximization thesis or more specifically to the well-defined utility function. [Silberberg and Suen 2001??]

Integrability

Suppose an econometrician estimates a set of demand functions $y_i^e = f_i(\mathbf{p}, M)$, i = 1, …, n, and asks you, the high-powered economist, to check whether these estimated functions are capable of being derived from the conventional utility function. To answer this problem, let us consider the following steps. First, given the demand function $y_i^e = f_i(\mathbf{p}, M)$ and using the homogeneity property, it can be rewritten as $f_i(t\mathbf{p}, tM) = f_i(\mathbf{p}, M)$. For the second step, let $t = 1/M$; thus the demand function can be rewritten as $f_i(\mathbf{p}, M) \equiv f_i(\mathbf{p}/M, 1) \equiv g_i(\mathbf{r})$ where $r_i = p_i/M$, i = 1, …, n and represents that fraction of a consumer's income necessary for the purchase of one unit of y_i^e. The third step is that in general we can expect the jacobian matrix of the g_i relative-price demand functions to have a nonzero determinant and to be able to solve for these relative prices in terms of y_i^e or $r_i = h_i(\mathbf{y})$. That is, we assume that $[\partial g_i/\partial r_j]$ is not equal to zero for all i,j = 1, …, n. Therefore, we can take the following system of equations $\partial y_i^e/\partial r_i \equiv \partial g_i/\partial r_i \equiv 0$ for all i = 1, …, n and solve for these relative prices in terms of the y_i^e or $r_i = h_i(\mathbf{y})$. The fourth step is that since $p_i/p_j = r_i/r_j = (p_i/M)/(p_j/M)$ and in equilibrium $dx_i/dx_j = -p_j/p_I$ (or $MRS_{ij} = -p_j/p_I$). Therefore $dx_i/dx_j = -r_j/r_i = h_j(\mathbf{y})/h_i(\mathbf{y})$ for each i,j = 1, …, n. The final step is to rewrite the fourth step as the following differential equation: $h_1(\mathbf{y})dy_1 + \ldots + h_n(\mathbf{y})dy_n = 0$. Solving the differential equation via integration we arrive at a utility function $U = \mu(\mathbf{y})$, such that $\partial\mu(\mathbf{y})/\partial y_i = h_i$ for i = 1, …, n. However, it must be noted that a solution to the above differential equation beyond the two-good (variable) case does not generally exist because it cannot be assumed that $\partial h_i/\partial y_j = \partial h_j/\partial y_i$. Therefore to eliminate this problem and ensure integrability, we have to impose an additional condition on the demand function, namely $\partial y_i^e/\partial p_j = \partial y_j^e/\partial p_i$ (see Silberberg and Weun, 2001, pages 325 – 332 for further discussion). So as in the case of revealed preference, integrability of empirically derived demand functions can only occur if we assume that the 'real world' is logically equivalent to the utility function.[21] [Silberberg and Suen 2001??] [More work here—say why it is a theoretical dead end]

Consumer Choice, Risk, and von Neumann-Morgenstern Utility Index

Lancaster's New Approach to Consumer Theory

[21] Footnote on whether this is really knowledge.

CHAPTER 9

CRITICISMS

Because the structure of neoclassical demand theory is hierarchical, a critique of it must start with preferences and utility functions, then proceed to the consumer demand curve, and finally to the market demand curve and end with reveal preference.[22] [More discussion here]

Preferences and the Utility Function

The criticism of neoclassical demand theory starts with examining preferences and the utility function. Thus, let us start with a consumer utility function of the general form:

(1) $U = \mu(\mathbf{y})$

where the vector of goods and services $\mathbf{y} = (y_1,\ldots, y_n) \geq 0$ and divisible. It is assumed by neoclassical economists that the individual consumer has preferences regarding each y_i, but, in general, they are not concerned how the consumer acquires them, that is, they are exogenously given in that the forces that produced them have no further impact. However, preferences have to come from somewhere, such as the consumer's family when s/he was a small child, since the consumer must have some social basis for identifying objects to have preferences about and socially derived reasons for preferring or not preferring y_i itself or relative to say y_j in the context of achieving a valued end.[23] Consequently, an individual consumer outside of a social network wanting y_i as a non-cultural object for its own sake is simply unintelligible. This argument implies that objects which consumers have preferences for are socially understood and hence have social characteristics that cannot be derived from their 'technical'

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[22] Since most special topics, such as duality, indirect utility functions, cost functions, and integrability depend on the existence of utility functions and demand curves, they cannot survive independently of this critique.

[23] It should be noted that the implicit or explicit definition of neoclassical economics includes the social embeddedness of all consumers and hence the social nature of their preferences.

characteristics.[24] Since the socially embedded consumer must have social preferences in order to make choices among socially understood goods and services that would achieve a valued end such as the maximizing of utility, then those preferences must be intrinsically non-autonomous because they are socially constructed.[25] More significantly there is no reason not to suppose that they are in part constructed and altered by the same industrial and social processes which the goods and services are produced to meet the valued ends desired by the consumer—that is, preferences are also endogenous. More strongly, it is plausible to argue that the 'social characteristic' of a good is constructed simultaneously with preferences, which means that neither can stand independently of the other. Hence a change in either means a change in both and if the social characteristic of the good also becomes vested in its price, then a change in price may have the Veblenian outcome of a change in both preferences and the good [reference needed]. This means that there can be no consumer or market demand curves for such a good, no price elasticity of demand, no possibility to talk about optimality of market equilibrium, and no possibility of an unchanging consumer. And it can also be plausibly argued that the latter result can be generalized in that the activity of social consumption generates a consumer with continuously changing preferences.[26] Such an outcome would also reproduce (as in Galbraith's revised sequence theory of demand) the consumer and her/his preferences that are the basis of making the choices. Hence, to initiate preference and demand theory by assuming that preferences are given

[24] There are also additional arguments that undermine Lancaster's new approach to consumer demand—see Watts and Gaston (1982-83).

[25] The socially embedded consumer with social preferences also has the capability of making interpersonal comparisons regarding consumption and other social activities (Peacock 1996). This, in part, undermines the theoretical core of neoclassical welfare economics—see Part VIII, chapters 36-37).

[26] Without the fixity of preferences, neoclassical welfare/cost-benefit arguments cease to have any meaning or substance—see Part VIII, chapters 36-37. [Check on Gintis on endogenous preferences]

relative to and independent of an array of given goods is to start the theory with nonsense which means that Robbins's definition of economics has no meaning [more].[27]

If preferences are socially constructed and articulated, then it is possible that the preference structure formation process or algorithm used by the consumer is also socially produced and the preference structure arising therefrom might not result in choices generating a unique utility maximizing outcome. To examine this point further, we shall assume, as in chapter 6, the axiom of comparability that a consumer can decide whether s/he prefers the vector of goods and services $\mathbf{y}^{\mathbf{i}}$ to $\mathbf{y}^{\mathbf{j}}$ or is indifferent them.[28] So for a consistent preference structure to exist that would permit maximization, the choice of vectors must be transitive or more generally acyclical so that it is not possible to have $\mathbf{y}^{\mathbf{1}}$ P(referred to) $\mathbf{y}^{\mathbf{2}}, \ldots, \mathbf{y}^{\mathbf{n-1}} \geq \mathbf{y}^{\mathbf{n}}$ and $\mathbf{y}^{\mathbf{n}} \geq \mathbf{y}^{\mathbf{1}}$. However, there is no apparent reason or possibility to restrict the possible social influences upon the consumer's choice making decisions since social influences are intrinsic to choice making decisions and are non-autonomous. Hence, it is quite plausible to conclude that the consumer relies on multiple influences when making decisions. But multiple influences easily generate

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[27] This conclusion raises severe doubts about Pareto efficiency in that the market does not act to adopt the given scarce resources to meet given ends; rather it is possible the market creates the ends to which it then allocates scarce resources. This possible outcome also renders incoherent the neoclassical definition of economics that is about making choices regarding scarce resources relative to given ends. [More discussion here]

[28] This assumption has a conceptual problem that can be called the "curse of dimensionality". For example, if it is assumed that there are thirty different goods and services and the quantity of each y_i can vary from zero to ten (although in principle the upper bound is unrestricted), the number of different $\mathbf{y}^{\mathbf{i}}$ would be 11^{30}. If each comparison of $\mathbf{y}^{\mathbf{i}}$ and $\mathbf{y}^{\mathbf{j}}$ took the consumer 1 billionth of a second, it would take her/him 5.53^{13} years to make all of them; and that period of time is not only longer than the life span of the consumer, it is also much longer than the known age of the universe. This example is rather crude relative to a more realistic example of a consumer making comparisons of goods and services vectors in a typical supermarket that has over 1000 different items; and in this case even if the quantities under consideration are zero or one, the time required to undertake all the comparisons would be even greater than the crude example. Hence, the axiom of comparability is simply incoherent, without any sense. It should be noted that the curse of dimensionality is distinct from radical uncertainty in that the latter rejects the possibility of comparisons because the consumer simply does not know about all goods and services that could be included in $\mathbf{y}^{\mathbf{i}}$ or all of the vectors of goods and services to be compared. Thus, if radical uncertainty also exists, the axiom of comparability simply ceases to be at all.

choices of vectors that are intransitive and/or cyclical as different influences are relevant when different vectors are compared; and without a single preference ranking of the vectors, the consumer's preference structure is inconsistent and therefore not a useful guide for utility maximization. Moreover, multiple influences combined with the "curse of dimensionality" implies that the consumer can rarely if ever attain a complete ordering of all the possible vectors of goods and services; and this also prevents the consumer's preference structure from being a useful guide for utility maximization.

Finally, since there are no restrictions on what the influences are, it is both plausible and possible that they

(1) produce a lexicographic preference structure that is transitive and acyclical and hence a consistent preference structure that is a guide for utility maximization; but such a preference structure violates the axiom of continuity and hence eliminates indifference curves;

(2) produce a fixed proportions (continuous or discreet) consumption patterns that are consistent with utility maximization but do not permit the derivation of the marginal utility of the individual goods involved; or

(3) result in the consumer adopting a frugal/green/non-materialist attitude that restricts consumption to a particular satisfactory or ecologically sustainable level or cultural/ethical/moral attitudes that affect choice decisions and consumption patterns independently of any utility consideration, hence resulting in decisions that are inconsistent with and/or not based on utility, utility maximization, and/or the axiom of dominance (or non-satiation).

In short, because the domain of influences is unrestricted and the curses of dimensionality and radical uncertainty ever present, it quite plausibly is not possible to exclude the consumer from having a preference structure that is incomplete, is without a single preference ranking, is in part lexicographic, contains fixed proportions consumption patterns, and is based on satiated, non-maximization choice decisions. Such a preference structure is inconsistent with a utility function that permits utility maximization, generates marginal utility (whether diminishing or not), and has indifference curves (whether strictly convex or not). In fact, it is plausible to suppose that such a preference structure is inconsistent with the concept of a utility *per se*. Just because consumers choose, this does not allow one to conclude that their choice decisions are consistent with utility functions *per se*, a utility maximizing function, or a strictly quasi-concave utility function which is generally assumed in textbooks when constructing consumer demand curves.[29] [Steedman 1980; Rizvi 2001; Baker 1988a, 1988b; Petrick and Sheehan 2002; Katzner 2002; Lane, et. al. 1996]

Consumer Demand Curves

Without an appropriate structure of preferences underlying, for example, a strictly quasi-concave utility function, it is not possible to derive a consumer demand curve and any of its derivative properties. That is, let us assume a strictly quasi-concave utility function and the derivation of the demand curve as developed in chapters 6-7 above. However, if, as is quite possible, the utility function does not exist or exists but with properties noted in the previous section, then there would be no basis for utility maximization, marginal rate of substitution, and the utility maximizing consumer demand curve since the first and second order conditions depend on the existence of the marginal utility of individual goods and services. Moreover, since the *Slutsky equation*, the *compensated demand* curve, and the concepts of the *substitution effect* and the *income effect* are also based on marginal utility, bordered Hessian matrix, and indifference curves, they would not exist or have any meaning. Without both effects, it is not possible to establish any connection between y_i and its price (thus leaving the quantity demanded unexplained) which implies there is no functional relationship (whether it be negative or positive)

[29] This conclusion undermines bounded rationality because rationality as defined in neoclassical microeconomic theory is incoherent; and without rationality of neoclassical microeconomic theory there is no *bounded* rationality for there is nothing to be bounded. [MORE WORK HERE]

between y_i and its price—thus there is no "law of demand" whatever that law might be. That is, the non-existence of the consumer demand curve arises because, after considering multiple influences, the consumer's choice decisions in face of a budget constraint minimizes the influence of or is made independent of prices. The absence of the substitution and income effects, the consumer demand curve has the further consequence of undermining the concept of price elasticity of demand. Finally, The absence of the utility function, marginal utility, utility maximization, and the consumer demand curve also means that the concepts of Giffen good, income elasticity of demand, cross-price elasticity of demand, consumer surplus, and duality are meaningless; that the homogeneity and budget constraint/adding-up properties of the demand curve are irrelevant; that the problems of the incompatibility of Giffen goods and market-determined prices and of integrability are non-problems; and that revealed preference theory cannot be logically linked to utility functions and consumer demand curves derived there from.

Revealed Preference

It should also be noted that revealed preference theory is methodologically incoherent in its own right (Wong, 1978) and without empirical support (Sippel, 1997). [NEED TO WORK ON]

Market Demand Curve

As noted in chapter 7 above, the market demand curve is derived by aggregating across consumer demand curves and it is assumed to have the same properties as the individual consumer demand curve.[30] However, the conditions for exact linear (or representational) aggregation are strict: that each consumer has a homothetic utility function (which generates linear Engel curves) and that the

[30] Consistent aggregation requires that all consumers have perfect knowledge so that the prices in their demand functions are the same. However, if uncertainty exists and some prices vary among the consumers, then consistent aggregation is not possible. The issue of uncertainty and failed expectations also affects the budget constraint when the consumer's income is a function of the expected prices of its

homothetic utility function for each consumer is the same. If these conditions (which produce consumer demand curves with all the right properties) do not hold, then the aggregate market demand curve that is derived has, aside from continuity and homogeneity, none of the properties of a consumer demand curve:

> …the aggregate [market] demand function will in general possess no interesting properties other than homogeneity and continuity. Hence, the theory of the consumer places no restrictions on aggregate [market] behavior in general. [Varian, 1992, p. 153]

In particular, there is no functional relationship between y_i and its price (so no law of market demand); and no aggregate (or market) versions of the substitution and income effects, price elasticity of demand, cross-price elasticity of demand, or the strong axiom of revealed preference theory.

Let us look at this more closely. The problem being considered is under what conditions can individual consumer demand curves be aggregated to give a market demand curve that behaves as if it represented the choices of a single utility maximizing consumer. That is, the market demand curve is conceived as qualitatively identical to consumer demand curve in that it is based on 'market' indifference curves and changes in quantity demanded due to changes in the market demand price depends both on *income* and *substitution effects*. However such equality between the market and consumer demand curve exists only if the indifference curves of each consumer in the market can be aggregated into "consistent" market indifference curves.

To deal with the problem, the conditions under which consumer indifference curves can be aggregated into a unique and consistent market indifference map have to be considered first.[31] Let us consider two consumers each with their own indifference map consisting of the same two goods:

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endowments, which means that the derivation of the consumer's demand curve is problematical. [Katzner, 1991]

[31] This section needs to be worked on.

Consumer A: $U_0^A = \mu_0^A(y_{1A}^0, y_{2A}^0)$; $U_1^A = \mu_1^A(y_{1A}^1, y_{2A}^1)$; ….

Consumer B: $U_0^B = \mu_0^B(y_{1B}^0, y_{2B}^0)$; $U_1^B = \mu_1^B(y_{1B}^1, y_{2B}^1)$; ….

The problem is whether a consistent market indifference curve map can be constructed from these consumer indifference curves: that is, does $U^m = \mu^m(U^A, U^B) = \mu^m(y_{1A} + y_{1B}, y_{2A} + y_{2B})$? Consider the following situation: in equilibrium for consumer A and B and for the market we have

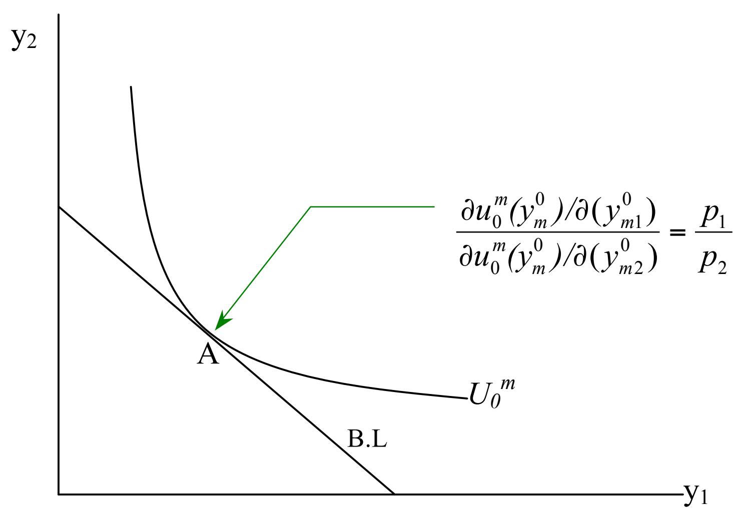
(a) $$\frac{\partial\mu_0^A(\mathbf{y_A^0})/\partial y_{1A}^0}{\partial\mu_0^A(\mathbf{y_A^0})/\partial y_{2A}^0} = \frac{p_1}{p_2}$$

(b) $$\frac{\partial\mu_0^B(\mathbf{y_B^0})/\partial y_{1B}^0}{\partial\mu_0^B(\mathbf{y_B^0})/\partial y_{2B}^0} = \frac{p_1}{p_2}$$

(c) $$\frac{\partial\mu_0^m(y_{1A}^0 + y_{1B}^0, y_{2A}^0 + y_{2B}^0)/\partial(y_{1A}^0 + y_{1B}^0)}{\partial\mu_0^m(y_{1A}^0 + y_{1B}^0, y_{2A}^0 + y_{2B}^0)/\partial(y_{2A}^0 + y_{2B}^0)} = \frac{p_1}{p_2}$$

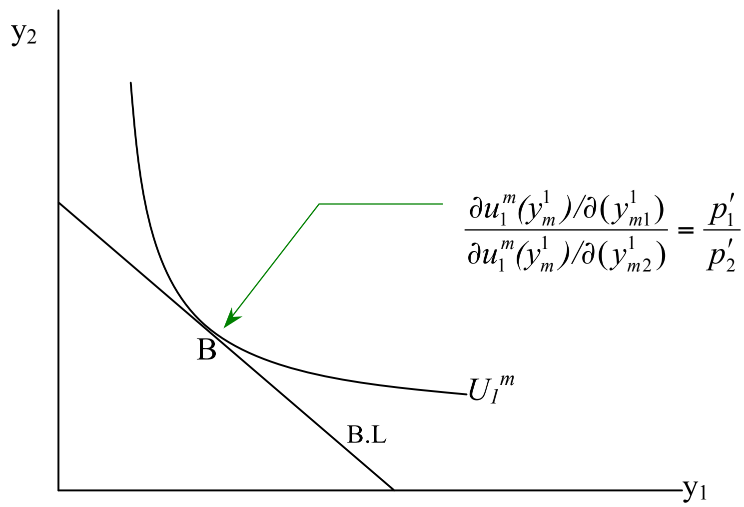
The question is whether for the same p_1 and p_2, are (a) and (b) equivalent to (c)? In general the answer is no. This can be seen in the following manner. First graph the above market equilibrium position:

Figure 9.1



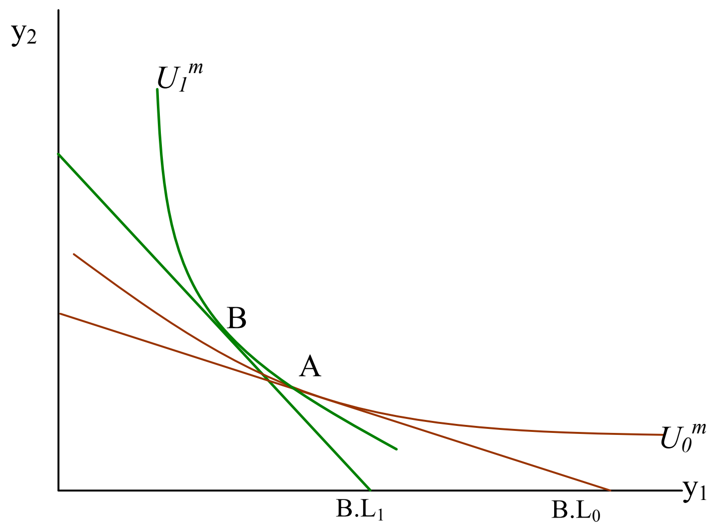
Now let us consider a new price ratio p_1'/p_2', the corresponding marginal rate of substitution for the market, and the new market indifference curve associated with the equilibrium position:

Figure 9.2



Now let us bring these two figures together:

Figure 9.3



[Work on]

That is the same y_{mA} and y_{mB} can be on two different market indifference curves or more directly, market indifference curves can cross.

The reason for this can be found in the determinants of y_{m1}^e and y_{m2}^e, the equilibrium quantities of goods one and two demanded in the market. That is, let $y_{m1}^e = y_{1A}^e + y_{1B}^e$ where $y_{1A}^e = f_1^A(p_1, p_2, M_A)$ and $y_{1B}^e = f_1^B(p_1, p_2, M_B)$. Consequently y_{m1}^e can be rewritten as $y_{m1}^e = f_1^e(p_1, p_2, M_A + M_B = M)$. As shown above, the only difference between y_{1A}^e and y_{1B}^e with respect to changes in p_1 arise from the different endowments of money income. Thus, the possibility arises that different distributions of income could produce the same y_{1m}^e (and y_{2m}^e) for different prices of p_1 (and p_2) or could produce a combination of y_{1m}^e and y_{2m}^e that lies on different market indifference curves.

To overcome this problem, the influence of the non-uniform distribution of money income among the consumers has to be eliminated. This occurs when each consumer in the market of good i have a linear Engel curve for that good and that each curve has the same slope. Consequently, the marginal propensity to consume for each consumer is independent of the level of income, implying that variations in p_i will not affect the consumption pattern of any consumer in the market. This also implies that each consumer has the *same homothetic utility function*. Consequently, the market demand function for good i, $y_{im}^e = f_i^m(\mathbf{p}, M)$ is a magnified version of any consumer demand function. In addition, the market demand curve can be obtained by summing horizontally the consumer demand curves.

Some neoclassical economists have attempted to avoid this outcome of no market demand curve by assuming (like Marshall) a "representative consumer" or just assuming that all consumers have the same homothetic utility function. But such assumptions are unjustified because they restrict what in principle cannot be restricted, which are the array of possible social influences upon consumer's choice making. Others have sought to reject aggregation and simply base the market demand curve on market price-quantity data. This implies, however, that neoclassical consumer preference and demand theory is irrelevant for understanding market activity. These responses are themselves dead ends if there are no utility functions (homothetic or not) or consumer demand curves (since with respect to the latter argument there would be no reason to presume any functional relationship between y_i and its price). In short, the conclusion must be that there is no basis for the existence of a market demand curve *per se*. [Rizvi, 1994 and 1998; Katzner, 1991; Varian, 1992; Mas-Colell, Whinston, and Green, 1995; and Deaton and Muellbauer, 1999]

Income and the Demand Curve

Income, therefore, seems to place a rather disrupting role in neoclassical demand theory. This appears to be due to the interrelationships between wants and activities as mediated by money income. That is, Marshall pointed out that activities and wants become interdependent once the increase in income has made possible the replacement of natural wants by civilized wants. However, modern demand theorists implicitly smuggled in this interrelationship into the demand curve in terms money income and the income effect. Now it is possible for variations in income to affect the propensity with which the consumer consumes a good and, hence, for the activities to "affect" wants. The only case where variations in money income has no affect on wants or the consumer's consumption pattern is when the consumer has linear Engel curves for all the goods he consumes (or as with Marshall the marginal utility of money is assumed constant or all the consumers come from the same income class). More, strongly, it is only when income and the "income effect" has been effectively neutralized—that is when all consumers have the same homothetic utility function—can the essential features of neoclassical demand theory, such as the market demand curve, be discussed at all. [more]